Mathematical Tables

and other

Aids to Computation

A Quarterly Journal

Edited by

V • Number 33 • January, 1951 • p. 1–56

Published by

THE NATIONAL RESEARCH COUNCIL

Washington, D. C.

NATIONAL RESEARCH COUNCIL

Division of Mathematical and Physical Sciences

SUBSCRIPTION RATES

1943–1945: (Nos. 1–12) $12.00 for all 12 issues (not available for separate sale except as noted below*)

1946–1949: $4.00 per year

1950–1951: $5.00 per year

Single issues are available for sale as follows:

* 1944 (No. 7, "A Guide to Tables of Bessel Functions," by H. Bateman and R. C. Archibald, 104 pp.) $2.00

1946–1949 (Nos. 13–28) $1.25 for single issue

1950–1951 (Nos. 29–36) $1.50 for single issue

All payments are to be made to National Academy of Sciences, 2101 Constitution Avenue, Washington, D. C.

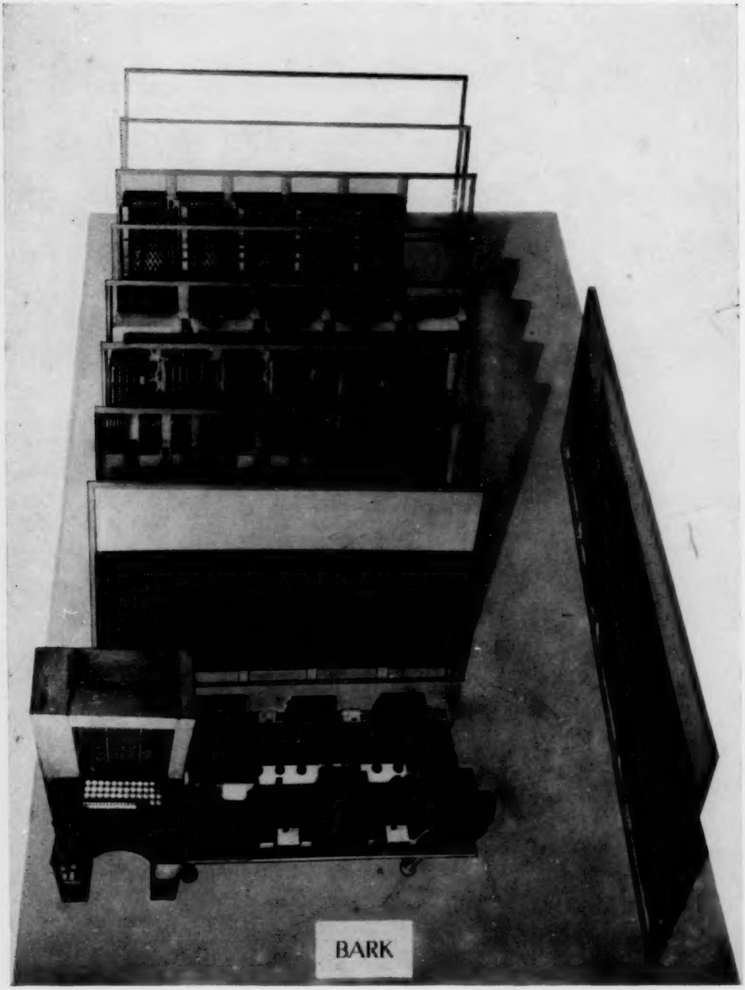
Agents for Great Britain and Ireland (subscription 42s, 6d for 1951) Scientific Computing Service, Ltd., 23 Bedford Square, London W.C.1.

Published quarterly in January, April, July and October by the National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation*, and all Books for review, should be addressed to Professor D. H. Lehmer, 942 Hilldale Ave., Berkeley, Calif.

Entered as second-class matter July 29, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.

).C.
the-
Calif.
ther
mer-
eley,
arate
a and
Con-
Scien-
ouncil,
Compu-
Hilldale
vania,


BARK

High Speed Sampling[1]

1. Introduction. Within the last ten years several electronic digital computing devices have been constructed or have reached advanced stages of development, e.g., the ENIAC, EDVAC, UNIVAC, SSEC, SEAC, SWAC, and a device at the Institute for Advanced Study. All these machines will perform additions, subtractions, multiplications, and divisions at high speeds. At least one of the devices can add two numbers in about one millionth of a second.

It would be desirable to introduce these high speed operations into "synthetic" sampling in statistics. At present such sampling may involve very time-consuming procedures. For example, suppose one wishes to estimate the distribution of a function (say, the standard deviation) of a random sample of size 50 from a population having the frequency function $f(x)$. ($f(x)$ may be troublesome to work with analytically; e.g., it might be an integral that is difficult to evaluate (see G in section 4).) An exact or even approximate mathematical formula for the sampling distribution may be impossible to calculate. However, the estimate can be obtained by drawing a large number of samples of size 50, computing the standard deviation of each sample, and making a frequency distribution of the standard deviations. In such work a machine for carrying out the sampling at high speed would be useful in conjunction with a high speed computing machine.

A high speed sampling machine (HSSM) might be such that directly from any given distribution[2] function, $F(x)$, a large number of random samples of a given size would be drawn. In this case standard high speed computation could then be carried out on each of the samples. Research is needed in regard to specifying physically a distribution function, $F(x)$, and in regard to sampling at high speed from a physically specified $F(x)$. Such phenomena as radioactivity, thermionic flow, and distributions of particle velocities may provide means of solving these problems. Unless the sampling machine's fundamental selection would be directly from $F(x)$, operations with the selected values would be necessary to effect random selections from $F(x)$.

A second possibility is that from a standard distribution function (say, the standard uniform distribution), a HSSM would draw a large number of samples of given size and by means of a high speed computing unit would transform each into a sample from any preassigned distribution function.

A third possibility is that a HSSM (by means of special high speed acceptance-rejection procedures regarding randomly selected values from a standard distribution function) would effectively draw random samples directly from a preassigned distribution function. Professor John von Neumann pointed out and discussed this possibility at a conference August 30, 1948.[1] The nature of the acceptance-rejection procedures is discussed in section 4 of this paper.

At present it seems that a HSSM of the second or of the third type would be easier to develop than one of the first type.

The purposes of this paper are: (1) to give brief descriptions of certain procedures for selecting numbers at random and for applying acceptance-rejection criteria to the numbers to obtain random sampling distributions;

(2) to give arguments for the validity of the interpretations of the procedures. The notation and other preliminary matters are given in section 2. The procedures, interpretations, and derivations are given in sections 3 and 4.

2. Notation and Preliminary Remarks.

A chance quantity, Y, having only two possible values (say 0 and 1) will be called a "random binary digit" when the following conditions are fulfilled precisely or to a close approximation:

$$\Pr(Y = 0) = \tfrac{1}{2}; \qquad \Pr(Y = 1) = \tfrac{1}{2}. \tag{2.1}$$

Let $Y_1, Y_2, \cdots, Y_c$ be c independent random binary digits. Let

$$Z = 2^{-1}Y_1 + 2^{-2}Y_2 + \cdots + 2^{-c}Y_c + 2^{-c-1}; \tag{2.2}$$

the set of possible values of Z consists of: 2^{-c-1}, $3\cdot 2^{-c-1}$, $5\cdot 2^{-c-1}$, $\cdots$, $(2^{c+1} - 1)\cdot 2^{-c-1}$. From (2.1) and (2.2) it follows that

$$\Pr(Z = (j + \tfrac{1}{2})2^{-c}) = 2^{-c}, \qquad (j = 0, 1, \cdots, 2^c - 1). \tag{2.3}$$

As c becomes larger, the probability distribution of Z is specified more and more closely by the standard uniform density function

$$f(z) = 1, \qquad (0 \leq z \leq 1). \tag{2.4}$$

In many situations Z could be considered for practical purposes as having the density function given in (2.4) if the value of c was not less than 10, say. The definition of Z in (2.2) indicates explicitly how random binary digits can be used to select at random a value of a chance quantity having approximately the standard uniform density function.

A chance quantity, S, having only ten possible values 0, 1, $\cdots$, 9 will be called a random decimal digit when the following conditions are fulfilled precisely or very closely:

$$\Pr(S = j) = \tfrac{1}{10}, \qquad (j = 0, 1, \cdots, 9). \tag{2.5}$$

The chance quantity

$$W = S_1 10^{n-1} + S_2 10^{n-2} + \cdots + S_n, \tag{2.6}$$

where $S_1, S_2, \cdots, S_n$ are mutually independent random decimal digits, will be called a random n-digit decimal integer. A random n-digit binary integer could be defined similarly.

Throughout this report a chance quantity will be represented by a capital letter and the variable of the chance quantity's distribution function will be represented by the corresponding small letter.

Let V be a chance quantity whose cumulative distribution function, $F(v)$, is continuous and monotone increasing. By definition

$$\Pr(V \leq v) = F(v). \tag{2.7}$$

It can be shown easily that the distribution function, say $Q(x)$, of $X = F(V)$ is

$$Q(x) = \Pr(X \leq x) = x, \qquad (0 \leq x \leq 1); \tag{2.8}$$

thus the density function, say $q(x)$, of X is

$$q(x) = 1, \qquad (0 \leq x \leq 1). \tag{2.9}$$

It is interesting also to consider a certain transformation from X to V. Owing to its special properties, the function $F(v)$ has an inverse, say $H(x)$, which is such that $F(H(x)) = x$. If X has the standard uniform distribution, then $V = H(X)$ has the distribution $F(v)$. It follows that if V's distribution is $F(v)$ and if x_0 is a randomly selected value of X, then $v_0 = H(x_0)$ is a randomly selected value of V. A machine that could at high speed obtain independently randomly selected values $x_1, x_2, \cdots, x_b$ of X, then perform the transformations $v_i = H(x_i)$ $(i = 1, \cdots, b)$, and then carry out standard, high speed computations with the v's would be a high speed sampling machine (see the fourth paragraph of section 1).

3. Selecting Values of Random Numbers.[3]

A. *Use of Radioactivity.*

PROCEDURE. Use a radioactive substance having a high average emission rate, m, over a unit time interval. Note whether the number, N, of emissions for a given unit time interval is odd or even. Let a quantity Y take the value 1 or 0 according as N is odd or even.

INTERPRETATION. Y is a random binary digit when m is large.

ARGUMENT. Assume that N has a Poisson distribution,

$$p(n) = e^{-m}m^n/n!, \qquad (n = 0, 1, \cdots), \tag{3.1}$$

where m is the average emission rate. The quantity $Y = \frac{1}{2}[1 - (-1^N)]$. From (3.1) it follows that

$$\begin{aligned} \Pr(Y = 0) &= e^{-m} \sum_{\alpha=0}^{\infty} m^{2\alpha}/(2\alpha)! = e^{-m}[e^m + e^{-m}]/2 = \tfrac{1}{2}[1 + e^{-2m}]. \\ \Pr(Y = 1) &= \tfrac{1}{2}[1 - e^{-2m}]. \end{aligned} \tag{3.2}$$

The convergence of $\Pr(Y = 0)$ and $\Pr(Y = 1)$ to the limit $\frac{1}{2}$ as m becomes large is unusually fast.

B. *Use of Several Coin Tosses.*

PROCEDURE. Perform K coin tosses. Note whether N', the number of Heads obtained, is odd or even. Let Y be 1 or 0 according as N' is odd or even.

INTERPRETATION. Y is a random binary digit when K is large, if no coin is badly biased.

ARGUMENT. Let $p_i = \frac{1}{2} + \alpha_i$, say, be the probability of Head on the i-th toss $(i = 1, \cdots, K)$. Assume that the tosses are mutually independent. Let $Y = \frac{1}{2}[1 - (-1)^{N'}]$. It can be shown that

$$\begin{aligned} \Pr(Y = 0) &= \tfrac{1}{2} + (-2)^{K-1} \prod_{1}^{K} \alpha_i, \\ \Pr(Y = 1) &= \tfrac{1}{2} - (-2)^{K-1} \prod_{1}^{K} \alpha_i. \end{aligned} \tag{3.3}$$

The assertions in (3.3) are obviously true for $K = 1$. By mathematical induction on K the assertions can be shown to be true for all values of K. Assume that for each i, $|\alpha_i| < .2$, say; then for large K, Y is approximately a random binary digit since $|2^{K-1} \prod_{1}^{K} \alpha_i| < (.4)^K$.

C. *Use of Two Coin Tosses.*

PROCEDURE. Perform two coin tosses, C_1, C_2. Note results if and only if C_1 is Head and C_2 is Tail or C_1 is Tail and C_2 is Head; when this condition is fulfilled, note whether C_1 is Head. If C_1 is Head and C_2 is Tail, select the value 0; if the reverse holds, select the value 1.

INTERPRETATION. Y is precisely a random binary digit.

ARGUMENT. Assume that the probability, p, of Head is the same for both tosses ($0 < p < 1$). Assume that the two tosses are independent; then

$$\text{(3.4)} \quad \Pr(Y = 0) = \Pr(Y = 1) = [p(1-p)]/[p(1-p) + (1-p)p] = \tfrac{1}{2}.$$

D. *"Middle of the Square" Method for Generating "Pseudo-Random" Numbers.*

PROCEDURE. Form a sequence $x_0, x_1, \cdots, x_{N-1}$ of integers, where $x_j (j = 0, 1, \cdots, N-1)$ is an n-digit decimal integer, x_{j+1} consists of the successive n digits in the *middle* of the $(2n)$-digit number x_j^2, and x_0 is a number whose successive digits are, say, $0, 1, 2, \cdots, (n-1)$.

INTERPRETATION. Under certain conditions on N and n (discussed below) $x_0, x_1, \cdots, x_{N-1}$ can be considered as independently selected values of a random n-digit decimal integer (see (2.6)).

ARGUMENT. The argument given here will be mainly empirical, although an *a priori* argument, based on certain results in ergodic theory, could be used. It should be noted that $x_1, \cdots, x_{N-1}$ are completely determined by x_0 and x_0 is not necessarily chosen at random; nevertheless, experience (discussed below) indicates that the interpretation above is reasonable. Accordingly, $x_0, x_1, \cdots, x_{N-1}$ are not inappropriately termed values of "pseudo-random" numbers.

For fixed n the sequence $x_0, x_1, \cdots, x_{N-1}$ becomes periodic as $N \to \infty$; and there are reasons for expecting trouble for $N \doteq \sqrt{2} \cdot 10^{n/2}$. For $n = 7, 8, 10$ trouble actually arose[4] for smaller values of N (roughly, $N = \sqrt{2} \cdot 10^{n/2-1}$). The experience for $n = 8$, $N \doteq 700$ and for $n = 10$, $N \doteq 3{,}500$ has been satisfactory; it is based on various distribution and correlation tests. (For example, for the case $n = 10$ a χ^2-test was applied to the 10×10 frequency table of occurrence of the value k in the q-th digit of x_i for each x_i. Also, the correlation of the k-th digit of x_i with the q-th digit of x_{i+p} was evaluated for $k, q = 0, \cdots, 9$ and $p = 0, \cdots, 5$; the distribution of the 545 correlation coefficients was examined.) The experience for $n = 7$ was unsatisfactory.

E. *Congruential Methods for Generating Pseudo-Random Numbers.*

PROCEDURE. Form a sequence $x_0, x_1, \cdots$, as follows. Select at random any non-zero 8-digit (decimal) integer as x_0. To compute x_1 multiply x_0 by 23. Next remove the 9-th and 10-th digits of this product and subtract this two-digit number from the remaining 8-digit number. This difference

is x_1. The number x_2 is produced in the same way from x_1 that x_1 was produced from x_0.

INTERPRETATION. The pseudo-random numbers x_j are being produced according to the general law

$$x_j \equiv x_0 k^j \pmod{M},$$

where in the above $k = 23$ and $M = 10^8 + 1$. These special values are particularly well adapted to a 10-digit decimal machine. Although in all cases the x_j are periodic, the periods are extremely large; in the above examples the x's are periodic with period 5882352. The method is due to LEHMER.[8]

ARGUMENT. The argument here is again empirical. The standard tests applied to such numbers gave satisfactory results. The method avoids the difficulty of the "middle of the square" method in which the sequence may unexpectedly terminate in a sequence of zero terms. It is designed primarily for use with parallel machines like the ENIAC and the SSEC.

4. Acceptance-Rejection Methods of Obtaining Empirical Random Sampling Distributions.

In this section it is assumed that if by some procedure a machine can, at high speed, randomly select a value of a chance quantity, then the machine can repeat the selection process a large number of times at high speed, where the selections are mutually independent. In view of this assumption only the random selection of a single value of a chance quantity need be considered.

A. *Random Selection of a Value of a Purely Discrete Chance Quantity.*

Let W be a purely discrete chance quantity whose possible values are, say, $w_0, w_1, w_2, \cdots, w_c$, and let

$$\Pr(W = w_j) = p_j, \qquad (j = 0, \cdots, c), \tag{4.1}$$

where $\sum_0^c p_j = 1$. Divide the interval $(0 \leq x \leq 1)$ into $c + 1$ mutually exclusive and exhaustive intervals $I_j (j = 0, 1, \cdots, c)$, where the length of I_j equals p_j. Let x_0 be a randomly selected value of a chance quantity having the standard uniform distribution; associate with x_0 the index, j, of the interval I_j in which x_0 lies; then, the value w_j can be considered as a randomly selected value of W.

B. *Random Selection of a Value of $Y = \cos Z$, where Z has the Uniform Density Function $f(z) = (2\pi)^{-1}$ $(0 \leq z \leq 2\pi)$.*

PROCEDURE. Let X be a chance quantity having the density function $f(x) = \frac{1}{2}$, $(-1 \leq x \leq 1)$. Select two values of X independently at random. If the sum of squares of the values exceeds 1, reject both values and repeat the selection process until a pair of values is obtained that satisfies the condition. Let x_1 and x_2 be such a pair. Form the ratio $x_1/(x_1^2 + x_2^2)^{\frac{1}{2}}$.

INTERPRETATION. $y = x_1/(x_1^2 + x_2^2)^{\frac{1}{2}}$ is a randomly selected value of Y.

ARGUMENT. The conditional probability element of a pair of acceptable values is $[(1/4)/(\pi/4)]dx_1dx_2 = (1/\pi)dx_1dx_2$. Let $x_1 = \rho \cos z$, $x_2 = \rho \sin z$; thus $\rho = (x_1^2 + x_2^2)^{\frac{1}{2}}$, $\cos z = (x_1)/(x_1^2 + x_2^2)^{\frac{1}{2}}$. The joint probability ele-

ment of the two new chance quantities is $(1/\pi)\rho d\rho dz$; and the probability element of Z is

$$f(z)dz = \pi^{-1}\int_0^1 \rho d\rho dz = (2\pi)^{-1}dz, \qquad (0 \leq z \leq 2\pi). \tag{4.2}$$

C. ***Random Selection of a Value of $U = -\ln X$, where X has the Standard Uniform Distribution.***

PROCEDURE. Let x_0 and x_i be two values of X selected independently at random. If and only if $x_1 < x_0$, select independently a third value, x_2, of X at random; if and only if $x_1 + x_2 < x_0$, select a fourth value, x_3, of X at random; etc. Discontinue the procedure upon first obtaining a value, x_n, of X such that

$$x_1 + x_2 + \cdots, + x_n \geq x_0. \tag{4.3}$$

If n is an even number, reject all the values obtained and repeat the procedure until a sequence $x_0, x_1, \cdots, x_n$ satisfying the relation in (4.3) is obtained, where n is an odd number. Retain x_0; and note the number, t', of sequences formed to obtain the first acceptable sequence. Let $t = t' - 1$.

INTERPRETATION. $t + x_0$ is a randomly selected value of $U = -\ln x$.

ARGUMENT. Let r_j by the probability that the sum of the elements of a sample of j values of X will be not less than a given value $x_0 (0 \leq x_0 \leq 1)$. It can be shown easily that

$$r_j = 1 - (x_0)^j/j!, \qquad (j = 1, 2, \cdots). \tag{4.4}$$

For any given attempt let N be the number of values of X selected, excluding x_0, to satisfy (4.3). We have that $\Pr(N = n) = r_n - r_{n-1}$, $(n = 1, \cdots; r_0 = 0)$. The conditional probability that N is odd, given x_0, is

$$\Pr(N \text{ odd} \mid x_0) = (r_1 - r_0) + (r_3 - r_2) + \cdots = e^{-x_0}. \tag{4.5}$$

Let T' be the number of sequences formed to obtain the first acceptable sequence. Let X_0 be the initial chance quantity in the acceptable sequence; X_0 takes the value x_0 in (4.3) when (4.3) is an acceptable sequence. Let $T = T' - 1$. The probability that a given attempt will be unsuccessful is $\Pr(N \text{ even}) = 1 - \int_0^1 e^{-x_0}dx_0 = e^{-1}$; thus the joint probability element of T and X_0 is

$$e^{-t}\cdot e^{-x_0}dx_0. \tag{4.6}$$

This element is "mixed" since T is discrete and X_0 is continuous. (4.6) implies that the probability element of $U = T + X_0$ is

$$e^{-u}du, \tag{4.7}$$

which is the same as the probability element of $U = -\ln X$. (It can be shown that the condition $x_0 \geq x_1 \geq \cdots, \geq x_n$ can be used in place of (4.3).)

D. ***Selection at Random of a Value of X, where the Density Function of X is $f(x)$, $(0 \leq x \leq a)$, and there is a Maximum Value of $f(x)$.***

PROCEDURE. Let b be the maximum of $f(x)$. Let $g(x) = f(x)/b$. Let U, V be independent chance quantities having the standard uniform distribution. Let $Z = aU$. At random select values v and z of V and Z, respectively.

Retain z if and only if $v < g(z)$. Continue the procedure of selecting values of V and Z until an acceptable z is obtained.

INTERPRETATION. z is a randomly selected value of X.

ARGUMENT. The joint probability element of Z and V before application of the acceptance-rejection procedure is $a^{-1}dzdv$. Clearly

$$\Pr(V < g(Z)) = a^{-1} \int_0^a dz \int_0^{g(z)} dv = (ab)^{-1}; \tag{4.8}$$

therefore the joint probability element of Z and V given that $V < g(Z)$ is

$$f(z, v \mid v < g(z)) = bdzdv, \tag{4.9}$$

and so

$$f(z \mid v < g(z))dz = b\left(\int_0^{g(z)} dv\right) dz = bg(z)dz = f(z)dz. \tag{4.10}$$

E. *Random Selection of a Value of $W = e^{-X}$, $(0 \leq x \leq k)$, where X has a Uniform Density Function $f(x) = 1/k$.*

PROCEDURE. Let u, v be randomly selected values of U, V which are independent chance quantities each having the standard uniform distribution. Retain u if and only if $uv < e^{-k}$ and $u \geq e^{-k}$. Continue until an acceptable u is obtained.

INTERPRETATION. u is a randomly selected value of W.

ARGUMENT. The density function of W is

$$\begin{aligned} h(w) &= 1/(kw), && (e^{-k} \leq w \leq 1), \\ h(w) &= 0, && (0 \leq w < e^{-k}). \end{aligned} \tag{4.11}$$

The argument in **D** above may be applied, where $a = 1$, $b = e^k/k$. The inequality condition in **D**, that v be less than $g(z)$, becomes here that v be less than e^{-k}/u when $u \geq e^{-k}$ and that v be less than 0 when $u < e^{-k}$. The latter requirement means rejection of any $u < e^{-k}$; thus the application of **D** leads to the procedure of accepting u if and only if $u \geq e^{-k}$ and $uv < e^{-k}$.

F. *Random Selection of a Value of X, where the Density Function of X is $f(x)h(x)$, $(0 \leq x \leq a)$, and f, h have Maxima.*

PROCEDURE. Let b and c be the maxima of $f(x)$ and $h(x)$, respectively; let $g(x) = f(x)/b$, $p(x) = h(x)/c$. Let U, V, W be mutually independent chance quantities having the standard uniform distribution. Let $Z = aU$. At random select values of Z, V, W. Retain z if and only if $v < g(z)$ and $w < p(z)$. Continue the procedure of selecting values of Z, V, W until an acceptable z is obtained.

INTERPRETATION. z is a randomly selected value of X.

ARGUMENT. Assume that both $f(x)$ and $h(x)$ are non-negative. By an argument very similar to that used in **D** above it can be shown that the conditional probability element of Z given $V < g(Z)$ and $W < p(Z)$ is

$$(bc)g(z)p(z)dz = f(z)h(z)dz. \tag{4.12}$$

G. *Random Selection of a Value of X, where the Density Function of X is $f(x) = \int_0^a g(x, u)du$, $(0 \leq u \leq a)$, $(0 \leq x \leq b)$, and g has a Maximum.*

PROCEDURE. Let c be the maximum of $g(x, u)$, $(0 \leq u \leq a; 0 \leq x \leq b)$. Let $p(x, u) = g(x, u)/c$. Let T, V, W be independent chance quantities having the standard uniform distribution. Let $S = aT$ and $Y = bV$. At random select values of S, Y, and W. Retain y if and only if $w < p(y, s)$. Continue the procedure of selecting values of S, Y, W until an acceptable y is obtained.

INTERPRETATION. y is a randomly selected value of X.

ARGUMENT. Assume that $g(x, u) \geq 0$. Before application of the acceptance-rejection procedure the probability element of S, Y, W is

$$(ab)^{-1}dsdydw. \tag{4.13}$$

From (4.13) it follows that

$$\begin{aligned} \Pr(W < p(Y, S)) &= (ab)^{-1} \int_0^b \int_0^a p(y, s)dsdy \\ &= (abc)^{-1} \int_0^b \int_0^a g(y, s)dsdy \\ &= (abc)^{-1} \int_0^b f(y)dy = (abc)^{-1}; \end{aligned} \tag{4.14}$$

therefore, the conditional probability element of S and Y given $W < p(Y, S)$ is

$$f(s, y \mid w < p(y, s))dsdy = cp(y, s)dsdy, \tag{4.15}$$

and

$$\begin{aligned} f(y \mid w < p(y, s))dy &= \left(\int_0^a cp(y, s)ds\right)dy \\ &= \left(\int_0^a g(y, s)ds\right)dy = f(y)dy. \end{aligned} \tag{4.16}$$

D. F. VOTAW, JR.
J. A. RAFFERTY

School of Aviation Medicine
Randolph Field, Texas

[1] This paper was presented under the title, "High speed selection of values of chance quantities," at a meeting of the American Statistical Association Dec. 28, 1949 and is a revision of reports submitted to the USAF School of Aviation Medicine, Aug. and Sept., 1948. The authors wish to acknowledge the valuable help of Professor J. VON NEUMANN.

[2] For simplicity of discussion the only chance quantities considered will be one-dimensional.

[3] In connection with this topic we may cite the following references.

H. BURKE HORTON, "A method for obtaining random numbers," *Annals Math. Stat.*, v. 19, 1948, p. 81–85.

JOHN E. WALSH, "Concerning compound randomization in the binary system," *Annals Math. Stat.*, v. 20, 1949, p. 580–589.

JOHN E. WALSH, "On a method of obtaining random binary digits by flipping coins," unpublished manuscript, Project RAND, Santa Monica, Calif., July, 1948.

JOHN W. MAUCHLY, "Pseudo-random numbers," unpublished report, presented Dec. 29, 1949 to the American Statistical Association.

D. H. LEHMER, "Mathematical methods in large scale computing," unpublished address presented at Harvard University Symposium, Sept., 1949.

[4] In empirical studies conducted at Los Alamos.

Radix Table for Trigonometric Functions and Their Inverses to High Accuracy

1. Introduction.—By means of two algorithms, based upon the addition theorem for $\tan x$, and a radix table of $\arctan(m\cdot 10^{-n})$, $m = 1(1)9$, $n = 1(1)6$, to 20D, one has the means of calculating $\tan x (x < \pi/4)$ and $\arctan x$ to at least 18D from a one-page table. The other trigonometric functions and their inverses follow easily. All division operations may be done upon an ordinary 10-bank desk calculator, so that no intermediate written work is necessary.

2. To obtain tan x.—If $x < \frac{1}{4}\pi$, subtract from x the largest tabular value of $\arctan a_1$ that leaves a positive remainder x_1; from this remainder subtract the largest tabular value of $\arctan a_2$ that leaves a positive remainder x_2. Continue in this way until the remainder $x_6 = x - \sum_{i=1}^{6} \arctan a_i$ is reached. The a_i are all exact, each with a single significant digit, unless zero.

By evaluating in turn

$$t_6 = \tan x_6 \doteq x_6, \quad t_{i-1} = \frac{t_i + a_i}{1 - t_i a_i}, \quad i = 6, 5, 4, 3, 2, 1;$$

one can obtain $t_0 = t = \tan x$. An easier alternative is to evaluate in turn

$$\begin{aligned} a &= a_1 + a_2, \quad b = a_3 + a_4, \quad c = a_5 + a_6; \\ d &= 1 - a_1a_2, \quad e = 1 - a_3a_4, \quad f = 1 - a_5a_6; \\ A_1 &= ae + bd, \quad A_2 = de - ab, \\ B_1 &= c + fx_6, \quad B_2 = f - cx_6. \end{aligned}$$

Then $t = \tan x \doteq (A_1B_2 + A_2B_1)/(A_2B_2 - A_1B_1)$.

If $\frac{1}{4}\pi < x < \frac{1}{2}\pi$, evaluate $x' = \frac{1}{2}\pi - x = 1.57079\,63267\,94896\,61923 - x$; then $\tan x = 1/\tan x'$ to as many significant figures as are known to 18 decimals in x'.

Other functions are given by

$$\sin x = t(1 + t^2)^{-\frac{1}{2}}, \quad \cos x = (1 + t^2)^{-\frac{1}{2}}, \quad \cot x = 1/t,$$
$$\sec x = (1 + t^2)^{\frac{1}{2}}, \quad \operatorname{cosec} x = (1 + t^2)^{\frac{1}{2}}/t.$$

3. To obtain arctan t.—If $t < 1$, evaluate in turn

$t_1 = (t - a_1)/(1 + a_1t)$ where a_1 is the first decimal digit of t,

$t_2 = (t_1 - a_2)/(1 + a_2t_1)$ where a_2 is the second decimal digit of t_1, the first being zero, and so on, until t_6 is reached, using

$$t_{i+1} = (t_i - a_{i+1})/(1 + a_{i+1}t_i).$$

Then $\arctan t \doteq \sum_{i=1}^{6} \arctan a_i + t_6$.

If desired, two successive stages can be combined, thus

$$t_{i+2} = \frac{(1 - a_{i+1}a_{i+2})t_i - (a_{i+1} + a_{i+2})}{1 - a_{i+1}a_{i+2} + (a_{i+1} + a_{i+2})t_i}.$$

If $t > 1$, $\arctan t = \frac{1}{2}\pi - \arctan(1/t) = 1.57079\ 63267\ 94896\ 61923 - \arctan(1/t)$. Other functions are given by

$$\operatorname{arccot} x = \tfrac{1}{2}\pi - \arctan x, \qquad \arcsin x = \arctan\left(x(1-x^2)^{-\frac{1}{2}}\right),$$
$$\arccos x = \arctan\left((1-x^2)^{\frac{1}{2}}/x\right), \qquad \operatorname{arcsec} x = \arctan(x^2-1)^{\frac{1}{2}},$$
$$\operatorname{arccsc} x = \arctan(x^2-1)^{-\frac{1}{2}}.$$

TABLE OF ARCTAN a_i.

a_1	arctan a_1	a_4	arctan a_4
.1	.09966 86524 91162 02738	.0001	.00009 99999 99666 66667
.2	.19739 55598 49880 75837	.0002	.00019 99999 97333 33340
.3	.29145 67944 77867 09200	.0003	.00029 99999 91000 00049
.4	.38050 63771 12364 88630	.0004	.00039 99999 78666 66871
.5	.46364 76090 00806 11621	.0005	.00049 99999 58333 33958
.6	.54041 95002 70584 15544	.0006	.00059 99999 28000 01555
.7	.61072 59643 89208 61654	.0007	.00069 99998 85666 70028
.8	.67474 09422 23552 66306	.0008	.00079 99998 29333 39887
.9	.73281 51017 86506 59164	.0009	.00089 99997 57000 11810
a_2	arctan a_2	a_5	arctan a_5
.01	.00999 96666 86665 23821	.00001	.00000 99999 99999 66667
.02	.01999 73339 73150 53306	.00002	.00001 99999 99997 33333
.03	.02999 10048 56877 89968	.00003	.00002 99999 99991 00000
.04	.03997 86871 23290 04141	.00004	.00003 99999 99978 66667
.05	.04995 83957 21942 76141	.00005	.00004 99999 99958 33333
.06	.05992 81551 21207 88443	.00006	.00005 99999 99928 00000
.07	.06988 60016 34642 49929	.00007	.00006 99999 99885 66667
.08	.07982 99857 12237 31589	.00008	.00007 99999 99829 33333
.09	.08975 81741 89950 52315	.00009	.00008 99999 99757 00000
a_3	arctan a_3	a_6	arctan a_6
.001	.00099 99996 66666 86667	.000001	.00000 09999 99999 99967
.002	.00199 99973 33339 73332	.000002	.00000 19999 99999 99733
.003	.00299 99910 00048 59969	.000003	.00000 29999 99999 99100
.004	.00399 99786 66871 46433	.000004	.00000 39999 99999 97867
.005	.00499 99583 33958 32217	.000005	.00000 49999 99999 95833
.006	.00599 99280 01555 16001	.000006	.00000 59999 99999 92800
.007	.00699 98856 70027 94902	.000007	.00000 69999 99999 88567
.008	.00799 98293 39886 63376	.000008	.00000 79999 99999 82933
.009	.00899 97570 11809 11676	.000009	.00000 89999 99999 75700

4. Examples.

A. To obtain $\tan x$, for $x = .56548\ 66776\ 46162\ 78292$:

Here arctan .6 is seen to be the largest value of $\arctan a_1$ such that $x - \arctan a_1$ is positive. From

$x - \arctan .6 = x_1 = .02506\ 71773\ 75578\ 62748$, a_2 is seen to be .02. From

$x_1 - \arctan .02 = x_2 = .00506\ 98434\ 02428\ 09442$, a_3 is seen to be .005. From

$x_2 - \arctan .005 = x_3 = .00006\ 98850\ 68469\ 77225$, a_4 is seen to be 0, $x_3 = x_4$, and a_5 is seen to be .00006. From

$x_4 - \arctan .00006 = x_5 = .00000\ 98850\ 68541\ 77225$, a_6 is seen to be .000009. Finally

$x_5 - \arctan .000009 = x_6 = .00000\ 08850\ 68542\ 01525.$

$a = a_1 + a_2 = .62$, $b = a_3 + a_4 = .0050$, $c = a_5 + a_6 = .000069$,

$d = 1 - a_1a_2 = .988$, $e = 1 - a_3a_4 = 1$, $f = 1 - a_5a_6 = .99999\ 999946$;

$A_1 = ae + bd = .62494$, $A_2 = de - ab = .9849$,

$$B_1 = c + fx_6 = .00006\ 98850\ 68541\ 53731,$$
$$B_2 = f - cx_6 = .99999\ 99993\ 98930\ 27060.$$
$$A_1B_2 = .62493\ 99996\ 24367\ 48331, A_2B_1 = .00006\ 88298\ 04006\ 56010,$$
$$A_2B_2 = .98489\ 99994\ 08006\ 42351, A_1B_1 = .00004\ 36739\ 74734\ 34833,$$
$$A_1B_2 + A_2B_1 = .62500\ 88294\ 28374\ 04341,$$
$$A_2B_2 - A_1B_1 = .98485\ 63254\ 33272\ 07518,$$
$$t = \tan x \doteq \frac{A_1B_2 + A_2B_1}{A_2B_2 - A_1B_1} = .63461\ 92975\ 44148\ 10040,$$

which is correct to about 3 units in the 19th decimal place.

B. To obtain arctan t, for $t = .59139\ 83513\ 99471\ 09817$:

Since $a_1 = .5$, we have

$$t_1 = (t - a_1)/(1 + a_1t) = \frac{.09139\ 83513\ 99471\ 09817}{1.2956\ 99175\ 69973\ 55491},$$
$$\text{or}\quad t_1 = .07053\ 97928\ 11176\ 17252, a_2 = .07;$$
$$t_2 = (t_1 - a_2)/(1 + a_2t_1) = \frac{.00053\ 97928\ 11176\ 17252}{1.0049\ 37785\ 49678\ 23321},$$
$$\text{or}\quad t_2 = .00053\ 71405\ 26474\ 81117, a_3 = 0;$$
$$t_3 = t_2, a_4 = .0005;$$
$$t_4 = (t_3 - a_4)/(1 + a_4t_3) = \frac{.00003\ 71405\ 26474\ 81117}{1.0000\ 00268\ 57026\ 324},$$
$$\text{or}\quad t_4 = .00003\ 71405\ 16499\ 97288, a_5 = .00003;$$
$$t_5 = (t_4 - a_5)/(1 + a_5t_4) = \frac{.00000\ 71405\ 16499\ 97288}{1.0000\ 00001\ 11421\ 55},$$
$$\text{or}\quad t_5 = .00000\ 71405\ 16492\ 01681, a_6 = .000007;$$
$$t_6 = (t_5 - a_6)/(1 + a_6t_5) = \frac{.00000\ 01405\ 16492\ 01681}{1.0000\ 00000\ 04998\ 36},$$
$$\text{or}\quad t_6 = .00000\ 01405\ 16492\ 00979.$$

$$\arctan t = \arctan .5 + \arctan .07 + \arctan .0005 + \arctan .00003 + \arctan .000007 + .00000\ 01405\ 16492\ 00979,$$
$$\text{or}\quad \arctan t = .53407\ 07511\ 10264\ 85054,$$

which happens to be correct to 20 decimals.

H. E. Salzer

NBSCL

RECENT MATHEMATICAL TABLES

834[**E**].—L. Prandtl & F. Vandrey, "Fliessgesetze normal-zäher Stoffe im Rohr. Ein Beitrag zur Rheologie," *Zeit. angew. Math. Mech.*, v. 30, 1950, p. 169–174.

This article gives two tables of functions having to do with viscous flow. The function $\phi(a)$ is defined by

$$\phi(a) = 8 \sum_{n=1}^{\infty} n(2n + 1)a^{2n-2}/(2n + 2)!$$

or by

$$\phi(a) = 2\{(a^2 - 2a + 2)e^a - 4 + (a^2 + 2a + 2)e^{-a}\}a^{-4},$$

the latter definition being given incorrectly by the authors.

The first table gives 4S values of ϕ for $a = 0(.1)5(.2)10$.

The second table gives, to 3D, values of $\phi(a\xi)/\phi(a)$ for $\xi = 0(.2).6(.1)1$ and $a = 1(1)10(2)14$.

D. H. L.

835[F].—N. G. W. H. BEEGER, "On composite numbers n for which $a^{n-1} \equiv 1 \pmod{n}$ for every a prime to n," *Scripta Math.*, v. 16, 1950, p. 133–135.

According to a theorem of FERMAT if a is any integer then $a^n - a$ is divisible by n, when n is a prime. The converse is false, however, that is, there exist composite numbers n dividing $a^n - a$ for all a. The smallest such anomalous composite number is $561 = 3 \cdot 11 \cdot 17$ and any number of this sort must be the product of at least three primes. The author studies the case of $n = pqr$. If p is given, there exist only a finite number of q's and r's for which pqr is an anomalous composite number. For each prime $p \leqslant 43$, there is given a table of all possible q's and r's. There are in all 52 such numbers given.

D. H. L.

836[F].—A. GLODEN, "Résolution de la congruence $X^4 + 1 \equiv 0 \pmod{p^3}$ avec une table," *Euclides*, v. 10, 1950, p. 74.

The table gives two solutions $<p^3/2$ of the congruence mentioned in the title for each prime $p < 200$. Since the congruence is solvable if and only if $p = 8n + 1$, the values of p considered are $p = 17, 41, 73, 89, 97, 113, 137,$ and 193.

D. H. L.

837[F].—A. VAN WIJNGAARDEN, "A table of partitions into two squares with an application to rational triangles," Nederl. Akad. Wetensch., *Proc.*, v. 53, 1950, p. 869–881 = *Indagationes Math.*, v. 12, 1950, p. 313–325.

The table, p. 872–881, gives all integers (x, y) such that $0 \leq x \leq y$ and

$$n = x^2 + y^2 \tag{1}$$

for each integer $n \leq 10000$ for which (1) has solutions.

The table extends the previously published table of BICKMORE & WESTERN[1] for $n = 1(1)1000$.

It is used to study the question of triangles with integral sides and rational medians, but is applicable to a number of other questions also.

The table was produced on a National machine, 12 columns at a time, by constant second difference procedure. Then the table was carefully rearranged. An alternative procedure would have been to use punched card equipment, in particular the summary punch and sorter.

D. H. L.

[1] C. E. BICKMORE & O. WESTERN, "A table of complex prime factors in the field of 8th roots of unity," *Messenger Math.*, v. 41, 1911, p. 52–64.

838[G].—H. W. BECKER, "Discussion of Problem 4277," *Amer. Math. Monthly*, v. 56, 1949, p. 697–699.

The author gives a table for $n = 1(1)25$ of N_n and N_n' respectively the number of non-commutative and commutative, non-associative products of n factors, and suggests the asymptotic formula

$$N'_{n+1} = .812k^n/(n + 1)\sqrt{n}$$

with $k = 2.48$.

J. RIORDAN

Bell Telephone Laboratories
New York, 14

839[G].—K. YAMAMOTO, "An asymptotic series for the number of three-line latin rectangles," Math. Soc. Japan, *Jn.*, v. 1, no. 4, 1949, p. 226–241.

KERAWALA's tables[1] for the number of reduced three-line latin rectangles, [that is, rectangles such that each line contains the numbers 1 to n, the first in natural order, and each column unlike numbers] for $n = 3(1)15$ is extended to $n = 15(1)20$. The last entry has 36 digits.

The reviewer, in his turn, by a new recurrence relation has carried this on to $n = 25$; the results are not yet published.

J. RIORDAN

[1] S. M. KERAWALA, "The enumeration of the latin rectangle of depth three by means of a difference equation," Calcutta Math. Soc. *Bull.*, v. 33, 1931, p. 119–127.

840[G].—R. ZURMÜHL, *Matrizen. Eine Darstellung für Ingenieure.* Berlin, 1950, Springer, xv, 427 p. 15.2 × 20.3 cm. Price 25.50 marks.

This book is a welcome addition to the literature in the field. Apart from a large number of numerical examples which illustrate the theories and their application, this book contains an extensive chapter (chapter VI) of 80 pages on numerical methods for the solution of linear systems of equations and the determination of the characteristic roots of finite matrices. It is with this chapter only that the present review is concerned. The author does not claim to present a complete enumeration of the known methods; those which are mentioned were chosen because of their suitability for engineering problems.

The methods are discussed in great detail and a complete work sheet is given many times.

For the solution of linear systems the author discusses on one hand the GAUSS elimination process and its more recent variations by CHOLESKY and BANACHIEWICZ and, on the other hand, some of the well known iteration processes. It is generally agreed to-day that the elimination process is the most convenient method, unless the system in question is of a special form which makes it more suitable for other methods, in particular if its matrix has a dominant principal diagonal. The matrix formulation employed in the Cholesky and Banachiewicz treatment makes it very suitable for calculating machines since some of the results can be obtained without recording all the intermediate steps. The related problem of inverting a matrix is also discussed.

The iteration processes have great drawbacks through convergence uncertainties. These methods have been studied extensively, e.g. in a now classical paper by VON MISES and POLLACZEK-GEIRINGER. There it is shown that for symmetric positive definite matrices the GAUSS-SEIDEL iteration process will converge always so that "normal" equations can, theoretically at least, be treated by it. This process, with error estimates by COLLATZ, is discussed as well as SOUTHWELL's relaxation method (which the author traces back to Gauss). The latter gave a method for checking as well as a device for improving convergence in difficult cases. This device does not seem to be known to modern relaxers. It consists in replacing the n unknowns $\bar{x}_i$ by a system of $n+1$ unknowns, x_0 and $x_i = \bar{x}_i - x_0, i = 1, 2, \cdots, n$. The coefficients of x_0 are given by $a_{i0} = -\sum_{k=1}^{n} a_{ik}$ $(i = 1, 2, \cdots, n)$ while the other coefficients are unaltered. An $(n+1)$-st equation is added: it is formed by the negative sum of the first n equations.

In the report of BODEWIG[1] it is stated that the combination of Gauss elimination and application of iteration afterwards is the most efficient method known so far. This was verified by the author in the case of 8 systems of 42 equations with small determinant.

Next, characteristic roots: the usual iteration process for the determination of the dominant root when it is real is discussed, particularly for the case of real symmetric matrices. Next, matrices with a complex dominant root are treated, both in the case of linear and non-linear elementary divisors. The latter case is based on the results of H. WIELANDT and for both cases the methods of DUNCAN and COLLAR are used. Collatz's "inclusion" theorem for the characteristic roots of real symmetric and positive (not necessarily symmetric) matrices is explained. It is a certain generalization of the iteration procedure for finding the dominant roots. Several methods are given for the determination of all the characteristic roots. Some of them were developed in Germany in recent years only and have not been published in easily accessible places before. There is the method (by KOCH) to apply the iteration process used to determine the dominant root by starting with a vector orthogonal to the vector that corresponds to the dominant root. Next the method of reducing the matrix to a matrix of one dimension less, but having the same roots as the original one apart from the dominant root. The methods of Duncan and Collar and of Wielandt are reported here. Wielandt's "fractional iteration" does not require the knowledge of the dominant root, but requires the knowledge of a suitable approximation to the next one. The process of Frazer, Duncan and Collar leads to the determination of the characteristic equation by applying powers of the matrix to an arbitrary vector. The method of HESSENBERG also leads to the characteristic polynomial by building it up from polynomials of lower degree.

O. TAUSSKY

NBS

[1] E. Bodewig, "Bericht über die verschiedenen Methoden zur Lösung eines Systems linearer Gleichungen mit reellen Koeffizienten I–V," Nederl. Akad. Wetensch., *Proc.*, v. 50, 1947, p. 930–941, 1104–1116, 1285–1295, v. 51, 1948, p. 53–64, 211–219 = *Indagationes Math.*, v. 9, p. 441–452, 518–530, 611–621, v. 10, p. 24–35, 82–90.

841[I, L].—G. BLANCH & R. SIEGEL, "Table of modified Bernoulli polynomials," NBS, *Jn. of Research*, v. 44, 1950, p. 103–107.

The polynomials to which the title refers may be defined by their Fourier series as follows

$$b_{k+1}(x) = -\sum_{n=1}^{\infty} n^{-k} \cos (nx + \tfrac{1}{2}\pi k)$$

and are related to the Bernoulli polynomials

$$B_k(x) = (B + x)^k$$

by the relation

$$2k!b_k(2\pi x) = (-2\pi)^k B_k(x)$$

so that $b_1(x) = (\pi - x)/2$, $b_2(x) = \dfrac{x^2}{4} - \dfrac{\pi x}{2} + \dfrac{\pi^2}{6}$, etc. The polynomials are given explicitly for $k = 1(1)11$ and $\left[x = 0\left(\dfrac{\pi}{36}\right)\pi;\ 17D\right]$. The values were computed from differences using the IBM 405 tabulator and checked by summation.

D. H. L.

842[I].—E. T. FRANKEL, "A calculus of figurate numbers and finite differences," *Amer. Math. Monthly*, v. 57, 1950, p. 14–25.

Figurate numbers are, effectively, taken as defined by generating functions

$$(1 - t)^{-n} = \sum F_r^n t^r$$

and thus are essentially binomial coefficients with sign convention reversed. Their relation to finite differences and sums depends essentially on the following results.

If $V(t) = \sum u_r t^r$ is the generating function of u_r $(r = 0, 1, \cdots)$, then $(1 - t)^{-1}V(t)$ is the generating function of $u_0 + u_1 + \cdots + u_r$ and $(1 - t)V(t)$ of $u_r - u_{r-1}$. The author writes $Su_r = u_0 + u_1 + \cdots + u_r$ and $S^{-1}u_r = u_r - u_{r-1}$ and defines their iterates in the usual way, which of course involves figurate numbers. The function generated by the product of two generating functions, now commonly called the convolution, he calls the criss-cross product. For n-th degree polynomials, special attention is given to numbers $S^{-(n+1)}u_r$, which the author calls d_r, because $d_r = 0$, $r > n$, and all other sums (or differences) of the given number sequence u_r can be expressed in terms of them. Other than illustrative tables, there are two main tables, one of figurate numbers F_r^n for $n = -7(1)7$ and $r = 0(1)7$ and one of $d_r = S^{-(n+1)}r^n$ for $n = 1(1)11$ and $r = 1(1)11$. The last have a long history (back to LAPLACE) and have lately been called cumulative numbers (DWYER), KUMMER numbers (PIZA), triangular permutation numbers (KAPLANSKY & RIORDAN).

J. RIORDAN

Bell Telephone Laboratories
New York 14, N. Y.

843[I].—E. Pflanz, "Allgemeine Differenzenausdrücke für die Ableitungen einer Funktion $y(x)$," *Zeit. angew. Math. Mech.*, v. 29, 1949, p. 379–381.

This paper gives an expression for the m-th derivative of a function $y(x)$ at a point x_0, i.e. $y^{(m)}(x_0)$, in terms of the functional values at $(n+1)$ points x_0 and $x_\rho \equiv x_0 + \alpha_\rho h$, $\rho = 1(1)n$. In general the points x_0, x_ρ may be spaced irregularly, but must be distinct. Also $y(x)$ is assumed to have a continuous $(n+1)$st derivative. The general formula is

$$y^{(m)}(x_0) = (-1)^m h^{-m} m! S_m y(x_0) - (-1)^n h^{-m} m! \prod_{\nu=1}^{n} \alpha_\nu \sum_{\rho=1}^{n} \alpha_\rho^{-m-1} \prod_{\nu=1}^{n}{}' (\alpha_\rho - \alpha_\nu)^{-1} F(m, n, \alpha_\rho) y(x_\rho) + R_{m,n} \qquad (1 \leq m \leq n).$$

Here $F(1, n, \alpha_\rho) = 1$ and

$$F(m, n, \alpha_\rho) = \sum_{\nu=0}^{m-1} (-1)^\nu \alpha_\rho^\nu S_\nu \qquad (2 \leq m \leq n).$$

S_λ denotes the sum of all possible products of λ distinct factors from the set $\alpha_1^{-1}, \alpha_2^{-1}, \cdots, \alpha_n^{-1}$ and the $\prod_{\nu=1}^{n}{}' (\alpha_\rho - \alpha_\nu)$ denotes the fact that the case $\nu = \rho$ is excluded. The remainder term $R_{m,n}$ is a rather involved expression which is given explicitly.

For $n + 1 = p + q + 1$ equi-distant points of interpolation, denoted by $x_0 + \rho h$, with $\rho = -p(1)q$, where p and q are integers $\geqslant 0$ and $p + q \geqslant 1$, the general formula is given in this special case. All formulas are given without proof, the only indication of their origin being the statement that they were obtained from Lagrange's interpolation formula. To facilitate the computations for equally spaced points x_ρ, the exact fractional values of S_λ are tabulated for $\lambda \leq p + q \leq 7$, $(p, q \geqslant 0)$, $\lambda = 1(1)7$.

The expression which is given for the derivatives for equally spaced points x_ρ, even with the author's auxiliary table of S_λ is far from being in the simplest form for computational purposes. The present article should be compared with a similar paper by Bickley,[1] which is omitted from the list of references. Bickley tabulates the exact integral quantities ${}_{mn}A_{pr}$ in the formula (retaining Bickley's notation)

$$n! w^m D^m y_p \doteq m! \sum_{r=0}^{n} {}_{mn}A_{pr} y_r$$

for $n = 2(1)6$, $m = 1(1)n$, $p = 0(1)n$, $r = 0(1)n$; for $n = 8, 10$, $m = 1(1)4$, $p = 0(1)n$, $r = 0(1)n$. Bickley also gives error terms. Although Bickley's formulas are much more direct and involve only a fraction of the computational work arising in the use of Pflanz's formulas, final judgment of the value of this note should be reserved until his expressions for the remainder may be compared with other forms given by Bickley and other writers in textbooks on finite differences.

In the heading of the table of $\sum_\lambda$ *for* $1 \leq p + q \leq 7$ *read* $\lambda \leq p + q \leq 7$.

H. E. Salzer

NBSCL

[1] W. G. Bickley, "Formulas for numerical differentiation," *Math. Gazette*, v. 25, 1941, p. 19–27.

844[K].—F. J. Anscombe, "Table of the hyperbolic transformation $\sinh^{-1} \sqrt{x}$," Roy. Stat. Soc., *Jn.*, A, v. 113, 1950, p. 228–229.

The function tabulated was proposed by the author[1] for use in transforming highly skewed distributions of counts to a more nearly normal form. In a forthcoming book on statistical methods[2] he also proposes its use in normalizing a variable obeying a Student-Fisher t-distribution. The tabulation values are given to 3D for $x = 0(.01)1(.1)10(1)200(10)500$.

C. C. C.

[1] F. J. Anscombe, "The transformation of Poisson, binomial and negative-binomial data," *Biometrika*, v. 35, 1948, p. 246–254.
[2] No title or publisher is given.

845[K].—W. G. Cochran & G. M. Cox, *Experimental Designs*. ix + 454 p. New York, Wiley & Sons, 1950. 15.8 × 23.7 cm. Price $5.75.

The tables presented are listings of plans for experimental designs and tables of random permutations of nine and of sixteen numbers.

Plan 4.1 gives one latin square arrangement each of sides 3(1)12, except that a set of four 4 × 4 squares is given. A randomization procedure is suggested which selects one square at random from all possible 3 × 3 squares or 4 × 4 squares. (Fisher & Yates[1] give complete representations up to 6 × 6 squares.) Plan 4.2 gives graeco-latin squares of orders 3, 4, 5, 7, 8, 9, 11 and 12.

Plans 6.1 and 6.2 present 2^3 and 2^4 factorial designs with the highest order interaction confounded in each and plan 6.3, a 2^6 factorial in 4 blocks with three four-factor interactions confounded. Plan 6.4 is a balanced group of partially confounded 2^4 factorial designs in blocks of 4 units, and plans 6.5 and 6.6, the same for 2^5 and 2^6 factorials in blocks of 8 units. Plan 6.7 gives a balanced group of four replications of a 3^3 factorial design in blocks of 9 units which partially confounds the highest order interaction (ABC), and plan 6.8, a balanced group for a 3^4 factorial which partially confounds each of the three-factor interactions. Plan 6.9 partially confounds ABC and BC of a 3×2^2 factorial design in blocks of 6 units. Plan 6.10 is a balanced group which partially confounds, for a 3×2^3 factorial, the two-factor and three-factor interactions involving two of the three factors at two levels each. This plan and the next also involve blocks of 6 units. Plan 6.11 is a balanced group for a $3^2 \times 2$ factorial, confounding partially AB and ABC. Plans 6.12 and 6.13 are balanced groups for a 4^2 factorial in blocks of 4 units and a 4×2^2 factorial in blocks of 8 units, both of which partially confound the highest order interaction. Plan 6.14 is a balanced group of nine replications of a 4 × 3 × 2 factorial in blocks of 12 units in which components of AC and ABC are partially confounded.

Plans 8.1a and 8.1b are 2^3 factorial designs in two 4 × 4 squares with AB, AC, BC and ABC partially confounded and with ABC completely confounded. Plan 8.2 is a 2^4 factorial in an 8 × 8 quasi-latin square with all three- and four-factor interactions confounded. Plan 8.3 gives a 2^5 factorial in an 8 × 8 quasi-latin square which completely confounds four of the four- and eight of the three-factor interactions. Plan 8.4 is a 2^6 factorial in five 8 × 8 quasi-latin squares so that each three- and four-factor interaction is confounded in two of the squares. Plan 8.5 gives a 3^3 factorial in two 9 × 9

quasi-latin squares with four of the degrees of freedom for ABC confounded in each. Plan 8.6 is a 3^4 factorial in two 9×9 quasi-latin squares in which all three-factor interactions are partially confounded. Plan 8.7 gives a 4×2^2 factorial in an 8×8 quasi-latin square with 2/3 confounding of ABC. Plan 8.8 is a $2 \times (2^2)$ factorial in a 4×4 half-plaid square with ABC completely confounded. Plan 8.9 gives a $2 \times (3 \times 2)$ factorial in a 6×6 half-plaid square with BC and ABC partially confounded. Plan 8.10 is a $3 \times (3 \times 2)$ factorial in two 6×6 half-plaid squares with AB and ABC partially confounded in each. Plans 8.11 and 8.12 are $2 \times (2^3)$ and $2 \times (2^4)$ factorials in 8×8 half-plaid squares with ABCD confounded in the first and ABD, BCE, ACDE, BCDE and ABCDE confounded in the second. Plans 8.13 and 8.14 are $2 \times 2 \times (2^3)$ and $2 \times 2 \times (2^4)$ factorials in 8×8 plaid squares with four and twelve third- and higher-ordered interactions confounded.

Plans 10.1 to 10.6 give $n \times n$ balanced lattice designs in n blocks and $n + 1$ replicates for $n = 3, 4, 5, 7, 8$ and 9. Plans 10.7 and 10.8 give the three replicates each for $n = 6$ and 10 to complete, with the first three sets of each of plans 10.1 to 10.6, the triple lattices. Plan 10.9 is a 12×12 quadruple lattice. Plans 10.10 to 10.16 give the three replicates of the $k \times (k + 1)$ rectangular lattices in blocks of k units for $k = 3(1)9$.

Plans 11.1 to 11.46 present in detail balanced incomplete block designs for all combinations (of treatments t, units per block k, replications r, blocks b, and λ, the number of times two treatments appear in the same block) that are considered by Fisher & Yates[1] with the exception of the $n \times n$ balanced lattices given in plans 10.1 to 10.6. There is included, as plan 11.5, one design *not* given by Fisher & Yates: (6, 3, 10, 20, 4).

Plans 12.1 to 12.8 are the $k \times k$ balanced lattice squares for $k = 3$ (prime powers) 13.

YOUDEN squares (incomplete latin squares) are given in plans 13.1 to 13.15 for $r < t$ in the following combinations:

t	7	7	11	11	13	13	15	15	16	16	19	19	21	31	37
r	3	4	5	6	4	9	7	8	6	10	9	10	5	6	9

Extensions of these squares for $r > t$ are given in plans 13.16 to 13.26 with

t	3	3	3	3	4	4	4	5	5	6	7
r	5	7	8	10	5	7	9	6	9	7	8

The relative efficiency of each design as compared with a randomized block layout is given, starting with plan 11.1.

Table 15.6 gives 1000 random permutations of the numbers 1(1)9 and table 15.7, 1000 random permutations of 1(1)16. The first table was constructed by reducing pairs of random digits modulo k and neglecting replicates of previously obtained residues. 200 permutations were obtained for each of $k = 9(1)13$ with numbers above 9 omitted. The authors do not point out a source of bias in that the digits 1 to 9 are not equally likely in this process. (The probability of residue = 1 is .1165.) The second table was also constructed by a mixture of methods but, this time, without bias. Sixteen pairs of random digits were ranked numerically with additional digits used to break ties in rankings. Permutations produced by the order of the ranks

gave 800 of the permutations; the remaining 200 were obtained by drawing from an urn. Both tables were tested for randomness by testing the distributions of numbers in positions and positions for numbers. A further test of the number of inversions in order in each permutation was made; none of the tests indicated significant deviation from random order.

LEO KATZ

Michigan State College
East Lansing, Michigan

[1] R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural, and Medical Research.* Edinburgh, 3rd ed., 1948 [*MTAC*, v. 3, p. 360–361].

846[K].—M. G. KENDALL, "Tables of autoregressive series," *Biometrika*, v. 36, 1949, p. 267–289.

The author gives further examples of autoregressive series calculated from $u_{t+2} + au_{t+1} + bu_t = \epsilon_{t+2}$, in which a and b are constants and ϵ is a random element, in addition to 4 he previously published.[1] There are 8 series of 400 terms each with ϵ rectangularly distributed with $a = -1.1$ in all 8 and $b = .5$ in the first 4 and .8 in the second 4. There are 5 series of 500 terms each constructed from $u_{t+1} - cu_t = \epsilon_{t+1}$ in which $c = .1(.2).9$ respectively and ϵ is normally distributed. The final set of 5 series of 500 terms each obeys the same difference equation as in the first 4 series but with the terms of each of the preceding 5 series taken as the values of the ϵ's in turn. The final set of 4 tables gives the product sums about zero, $\sum_t u_t u_{t+k}$ and estimates of the serial correlation to 3D for $k = 0(1)50$ for the first and for $k = 0(1)30$ for the remaining 3 of the 4 previously published series; to 4D and $k = 0(1)4$ (for the serial correlation) for the first 8 of the present series and also for the 2 series of 1600 terms obtained from each of the 2 sets of 4, and to 3D for $k = 0(1)30$ for the 5 series in which ϵ is normally distributed.

C. F. KOSSACK

Purdue University
Lafayette, Indiana

[1] M. G. KENDALL, *Contributions to the study of oscillatory time-series*, National Institute of Economic and Social Research. *Occasional Papers.* IX. Cambridge and New York, 1946.

847[K].—L. W. POLLAK, assisted by U. N. EGAN, *Eight-Place Supplement to Harmonic Analysis and Synthesis Schedules for Three to One Hundred Equidistant Values of Empiric Functions.* (Dublin Institute for Advanced Studies, School of Cosmic Physics, Geophysical Memoirs No. 1, Parts 1 and 2), Dublin, 1949. Parts 1 and 2, separately bound: xix, 43 p; 72 p., 23.8 × 32.6 cm, 7s 6d each.

This work is supplementary to the *Harmonic Analysis and Synthesis Schedules for Three to One Hundred Equidistant Values of Empiric Functions, by* L. W. POLLAK, assisted by C. HEILFRON [*MTAC*, v. 2, p. 306–307] and is intended to be used in conjunction with them or with the *All Term Guide for Harmonic Analysis and Synthesis using* 3 *to* 24; 26, 28, 30, 34, 36, 38, 42, 44, 46, 52, 60, 68, 76, 84 *and* 92 *equidistant values*, by L. W. POLLAK and U. N. EGAN, although it is also independently useful in harmonic analysis and synthesis. Its purpose is to provide more accurate values of sines, cosines, and angles pertinent to the harmonic analysis or synthesis of equidistant

values of empiric functions than were given in the publication to which it is supplemental.

For the preponderance of problems in harmonic analysis or synthesis, such as those commonly encountered in geophysics, economics, or ordinary physics laboratory work, the tables provided in the former publication are more than amply sufficient in accuracy. The present work, however, should be extremely valuable to astronomers and others having need for unusually high accuracy in computation, since angles are given to 10^{-12} degree and 10^{-10} second (although reliable to only 10^{-8} second), and sines and cosines to eight decimal places in one of the included tables, and to ten decimal places in another.

Part 1 consists of an "Introduction," and two tables called "Appendix" and "Register"; Part 2 consists of a table known as the "Index." The "Register" and "Index" contain the same information as the portions of the previous publication having the same captions, the difference being in the greater number of significant figures given for values in the later work.

The "Introduction" includes description of the method of computing the tabulated values, illustrated by three short tables, the reliability and accuracy of the values, and a brief discussion of the method of using the tables, either with or without the use of the previously published *Schedules*, or the *All Term Guide*. Three short additional tables illustrate suitable work forms and methods, respectively, for harmonic analysis and synthesis using the tables of this publication and the *Schedules*, and the construction of a schedule by means of this *Supplement* only. An extremely brief bibliography (7 items) is given.

The "Appendix" provides sine or cosine values to ten decimal places, listed according to the 2068 identification numbers ("T-numbers") given in the *Schedules*, which indicate the sine or cosine value to be used as a multiplying factor of each empiric value in harmonic analysis. Since the same values as multipliers are used repeatedly at various places in any series of empiric values, their identification by such serial T-numbers (that of their order of appearance in schedules for analysis of a series of n empiric values, n increasing from 3 to 100, sine or cosine values equal to 0, $\frac{1}{2}$ or 1 being so stated rather than identified by a serial T-number), is a convenience in saving labor and space. Pollak has previously published[1] a set of tables giving multiples of each of the first 120 "T-numbers." The sine or cosine values of the "Appendix" were computed by Pollak, using the angles to twelve decimals of a degree, as presented later in the "Index," and interpolating linearly, extending the interpolation to the second differences, with the ten-place values given in E. Engel's tables of sines, cosines, and tangents [*MTAC*, v. 1, p. 131, 170–171].

The "Register" presents values of angles in increasing order of magnitude, accompanied by their identifying T-number. These angles are given both in degrees, minutes, and seconds to the 10^{-10} second (reliable to only 10^{-8} second), and in degrees and decimal fractions of a degree to 10^{-12} degree. Each angle given in degrees and decimal parts of a degree was derived individually by dividing the product $360°i$ by n, the number of equidistant empiric values in a period. The last figure retained, in each case, was rounded off. Accuracy of these values was insured by the authors, using an inde-

pendent check process. The rounded off values thus obtained were then converted into minutes and seconds. Since the rounding-off process limits significance to $\pm 5 \cdot 10^{-12}$ degree, or 1.8×10^{-9} seconds, the presentation of values to 10^{-10} second in this table is misleading.

The "Index" presents the same angular values (iz), together with their complements ($\lambda z'$), and the multiplying factors $\sin iz = \cos \lambda z'$ given to the eighth decimal place, arranged in ascending magnitude of n, and also, consequently, of their identifying T-numbers, both of which are furnished. All the cosines and sines were computed twice, once by Miss NUALA O'BRIEN, using PETERS'[2] Eight-Place Tables of Trigonometric Functions, and independently by Pollak (the values given in the Appendix), and, in addition, a schedule-by-schedule check was made, so that the error is less than 5×10^{-9}.

The present work fills an apparent need for a set of such tables, giving sine, cosine, and angular values to greater than usual accuracy [cf. *MTAC*, v. 2, p. 307]. Because of the frequent practice of those having limited library funds, who usually purchase only one—the most accurate—set of a given type of tables, and therefore, in the present case, might choose to purchase only the *Supplement*, without the *Schedules*, it seems to the reviewer that repetition of much of the material included in the "Introduction" to the *Schedule* might have been advisable. This included an excellent, condensed, presentation of the procedures used in analysis and synthesis, for grouped data, non-equidistant values, mean values, and values having non-cyclic change, together with WALKER's short-cut method, and formulae for computation of error. It also seems that a much more comprehensive bibliography would have been appropriate.

The quality of printing and of the paper used in this *Supplement*, although somewhat inferior to that used for the *Schedules*, is good.

MARCELLA L. PHILLIPS

NBS

[1] L. W. POLLAK, *Rechentafeln zur Harmonischen Analyse*. Leipzig, 1926.

[2] J. PETERS, *Achtstellige Tafel der trigonometrischen Funktionen für jede Sexagesimalsekunde des Quadranten*. Berlin, 1939.

848[K].—J. WESTENBERG, "A tabulation of the median test for unequal samples," Nederl. Akad. Wetensch., *Proc.*, v. 53, 1950, p. 77–82, = *Indagationes Math.*, v. 12, 1950, p. 8–13.

Given that in a pooled sample of $N_1 + N_2$ from a univariate statistical universe $2n$ observed values of the variate do not and $N_1 + N_2 - 2n$ do exceed a given critical value, the author writes the joint probability that in the sample of $N_1, n + \Delta$ values and in the sample of $N_2, n - \Delta$ values do not exceed this value. The critical value is taken to be the median of the pooled sample. Then the number of values in the sample of N_1 lying between the median of this sample and the median of the pooled sample is $\frac{1}{2}(N_2 - N_1) + \Delta = \delta$. The author remarks that the above probability is maximal for $\delta = 0$ and is symmetrical with respect to N_1 and N_2. He then tabulates the values of $|\delta|$ to 1D that will be exceeded in such samples with probabilities .001, .005, .01(.01).05 for N_1 and N_2 = 6, 10, 20, 50, 100, 200, 500, 1000, 2000. The mode of calculation is not described but evidently $|\delta|$ is treated as a continuous variable whereas in actual samples $|\delta|$ is always an integer.

The experimenter can then tell from the table where his δ lies with respect to the given percentage points. A second table and logarithmic chart give for the same set of values of N_1 the minimal values of N_2 to 1D (A remark similar to that with regard to fractional values of δ could be made here.) for which the two samples could result in a $|\delta|$ at the significance levels given above. The author gave the results for $N_1 = N_2$ in a previous paper.[1]

C. C. C.

[1] J. WESTENBERG, "Significance tests for median and interquartile range in samples from continuous populations of any form," Nederl. Akad. Wetensch., *Proc.*, v. 51, 1948, p. 252–261.

849[K].—A. VAN WIJNGAARDEN, "Table of the cumulative symmetric binomial distribution," Nederl. Akad. Wetensch., *Proc.*, v. 53, 1950, p. 857–868, = *Indagationes Math.*, v. 12, 1950, p. 301–312.

The function tabled is

$$P(n, c) = 1 - 2^{1-n} \sum_{s=0}^{c} \binom{n}{s} = 2^{-n} \sum_{s=c+1}^{n-c-1} \binom{n}{s}$$

to 5D for $n = 1(1)200$ and $c = 0(1)\left[\frac{n-1}{2}\right]$. Noting that for n even

$$P\left(n, \frac{n-2}{2}\right) = 2^{-n}\binom{n}{n/2},$$

we see that the table enables one to find all the partial sums in $(\frac{1}{2} + \frac{1}{2})^n$ for $n \leq 200$. In order to save space the tabulation is made with the arguments c and $n - 2c$. Values of $P(n, c)$ to 7D for $n = 1(1)49$ are also available in the tables recently published by the National Bureau of Standards.[1] The reviewer compared the two tables for $n = 48$ and 49 as a check of the present author's assertion that his accuracy "is well under one unit of the last (fifth) decimal place." The differences were less than 1 unit in this place but for $n = 48$, $c = 9, 16$; $n = 49$, $c = 16, 17$ the final digit differed by unity from the rounded off value from the NBS tables. In the present tables no signs are indicated for final 5's.

C. C. C.

[1] NBS, *Tables of the Binomial Probability Distribution*. NBS Applied Math. Series, No. 6, Washington, 1950 [*MTAC*, v. 4, p. 208–209].

850[L].—M. ABRAMOWITZ, "Tables of integrals of Struve functions," *Jn. Math. Phys.*, v. 29, 1950, p. 49–51.

The standard notations[1] (p. 328–329) are used, and the integrals in question are

$$\bar{H}_n = \int_0^x H_n(t)dt, \qquad \bar{L}_n = \int_0^x L_n(t)dt.$$

$\bar{H}_0$, $\bar{H}_1$ are tabulated for $x = 0(.1)5$ to 6D and for $x = 5(.1)10$ to 5D; $\bar{L}_0$, $\bar{L}_1$ for $x = 0(.1)5$ to 6D, for $x = 5(.1)10$ to 6S. The tabulated values were obtained by numerical integration, using WATSON's tables[1] (p. 666–685) for $H_n(t)$ and tables of the Computation Laboratory of the NBS for $L_n(t)$. Spot checks were made by using the series expansion of the integrals in powers

of x. It is stated that in general, interpolation with five-point Lagrangean interpolation coefficients will yield the full accuracy of the tables.

The integrals arose in a paper by LEVINE & SCHWINGER.[2]

A. E.

[1] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*. Cambridge, 1922.
[2] H. LEVINE & J. SCHWINGER, "On the theory of diffraction by an infinite plane screen. I," *Phys. Rev.*, s. 2, v. 74, 1948, p. 958–974.

851[L].—S. CHANDRASEKHAR & G. MÜNCH, "On the integral equation governing the distribution of the true and the apparent rotational velocities of stars," *Astrophys. Jn.*, v. 111, 1950, p. 142–156.

The authors discuss the numerical solution of the integral equation

$$g(y) = y\int_y^\infty x^{-1}(x^2 - y^2)^{-\frac{1}{2}}f(x)dx. \tag{1}$$

The analytical solution involves differentiation and hence is not very reliable when $g(y)$ is given in the form of a histogram, and the authors hold that any direct numerical solution of (1) is likely to encounter the same difficulty. They advocate the use of either of two methods. (i) Determine $f(x)$ from its moments by means of the formula (given in the paper) for converting moments of g into moments of f. (ii) Assume a shape for $f(x)$ and determine the parameters by fitting the corresponding g to the observed $g(y)$.

In following up the second alternative, they use

$$f(x, x_1) = \pi^{-\frac{1}{2}}\{e^{-(x-x_1)^2} + e^{-(x+x_1)^2}\}.$$

For the corresponding $g(y, x_1)$ they give an integral representation, several expansions, and some numerical material.

Table 1(p. 148) gives

$$\bar{x} = \pi^{-\frac{1}{2}}e^{-x_1^2} + x_1\Phi(x_1) \text{ to 4S,} \quad \overline{x^2} = x_1^2 + \tfrac{1}{2} \text{ to 2D,}$$
$$\overline{x^3} = \pi^{-\frac{1}{2}}e^{-x_1^2}(1 + x_1^2) + (\tfrac{3}{2} + x_1^2)x_1\Phi(x_1) \text{ to 3S,}$$

and

$$\bar{x}/[2(\overline{x^2} - \bar{x}^2)]^{\frac{1}{2}} \text{ to 4 or 5S for } x_1 = 0(.1)1(.2)3.$$

Here

$$\Phi(x_1) = 2\pi^{-\frac{1}{2}}\int_0^{x_1} e^{-t^2}dt.$$

Table 2 (p. 150) gives $F_n(y)$ to 4 to 6D for $n = 0(1)6$ and $y = 0(.2)2.6$, where

$$F_n(y) = \pi^{-\frac{1}{2}}2^{2n+1}I_n(y)/(2n)!,$$
$$I_0(y) = \tfrac{1}{2}\pi\{1 - \Phi(y)\}, \qquad I_1(y) = \tfrac{1}{2}\pi^{-\frac{1}{2}}ye^{-y^2},$$

and

$$I_{n+1} = (n - \tfrac{1}{2} + y^2)I_n - (n - 1)y^2I_{n-1}, \qquad n = 1, 2, \cdots.$$

Table 3 (p. 153) gives

$$g(x_1, x_1) = \pi^{-\frac{1}{2}}x_1\int_{x_1}^\infty x^{-1}(x^2 - x_1^2)^{-\frac{1}{2}}e^{-(x-x_1)^2}dx \text{ to 4D}$$

for $x_1 = 2(.5)5(1)10$.

Table 4 contains the results of astronomical computations.

A. E.

852[L].—V. N. Faddeeva & M. K. Gavurin, *Tablitsy Funktsiĭ Besselia $J_n(x)$ tselykh nomerov ot* 0 *do* 120 [Tables of Bessel Functions $J_n(x)$ of integral order from 0 to 120]. Pod redaktsieĭ L. V. Kantorovicha. (*Matematicheskie Tablitsy*, no. 2.) Akad. Nauk, SSSR, Matematicheskiĭ Institut imeni V. A. Steklova, Moscow and Leningrad Gostekhizdat, 1950, 440 p. An errata slip containing 23 corrections is inserted in this volume. 17 × 26 cm. Cloth, 21 roubles. An edition of 4000 copies was issued in April.

The attractive appearance of this publication of the Academy of Science is in marked contrast to no. 1 of the series [RMT**854**]. It contains the following four tables:

Table 1, p. 9–371: $J_n(x)$ for $n = 0(1)120$, $x = [0(.1)124.9$; 6D, $\delta^2]$. In $J_{120}(x)$ the first significant value .000001 is for $x = 95.4$.
Table 2, p. 373–382: Zeros <125 of $J_n(x)$ to 5D. There are 40 zeros for $J_0(x)$, 39 for $J_1(x)$, and the last is a single zero for $J_{115}(x)$. Most of the values given here are new.
Table 3, p. 383–388: Interpolation coefficients.
Table 4, p. 389–439: $J_n(x)$, $x = [0(.01)14.99$; 8D], $n = 0(1)13$.

The authors state that it was not until their tables had been completed that they became acquainted with the first 8 volumes of the Harvard Bessel Function tables. With the twelfth and final volume of the Harvard tables we have at our disposal the values of $J_n(x)$, $n = 0(1)135$, $x \leq 100$, at interval $\leq .1$, to 10D at least.

For $x \leq 100$, all the values of $J_n(x)$ in the two new Russian tables are in the Harvard tables, which do not, however, give any explicitly stated zero values. Thus there is an appreciable amount of new results here. Not alone on account of difference in cost (less than 37 roubles as compared with $96) many workers will probably often find it convenient to turn to the two volumes of Russian tables, if it is found that they are reliable.

Since the Russians appear to have started the publication of a series of mathematical tables, already including two tables of Bessel functions, let us hope that the series will include the Tables of Bessel Functions with Complex Argument, announced in *Matematicheskiĭ Sbornik*, v. 51, 1941 [*MTAC*, v. 3, p. 66], but, as far as we know, never published.

R. C. Archibald

Brown University
Providence, R. I.

853[L].—C. W. Horton, "A short table of Struve functions and of some integrals involving Bessel and Struve functions," *Jn. Math. Phys.*, v. 29, 1950, p. 56–59.

J_n is the Bessel function, H_n the Struve function,

$$C_n = \int_0^z t^n J_n(t)dt, \qquad D_n = \int_0^z t^n H_n(t)dt,$$

Tables of J_n, H_0, H_1 are known.[1] The present paper gives H_n for $n = 2(1)4$, C_n for $n = 1(1)4$, and D_n for $n = 0(1)4$, all for $z = 0(.1)10$ mostly to 4D. Values of H_n have been computed to 7D by means of the power series and

the recurrence relations, and then rounded off to 4D because of the coarseness of the interval.

Values of C_0 were taken from a table by LOWAN & ABRAMOWITZ,[2] and values of C_n obtained by means of numerical integration and a recurrence relation. For $n = 3, 4$ and $z \geq 6$ only 3D are given.

The situation with regard to D_n is similar, except that here a table by J. W. WRENCH[3] was the point of departure.

It is believed that the maximum error is .6 units of the last decimal.

A. E.

[1] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*. Cambridge, 1922, p. 666–685.

[2] A. N. LOWAN & M. ABRAMOWITZ, *Jn. Math. Phys.*, v. 22, 1943, p. 3–12 [*MTAC*, v. 1, p. 154].

[3] *MTAC*, v. 3, p. 66.

854[L].—L. A. LIÛSTERNIK, I. ÎA. AKUSHSKIĬ & V. A. DITKIN, *Tablîtsy Besselevykh Funktsiĭ*. (*Matematicheskie Tablîtsy*, no. 1.) [Tables of Bessel Functions. (Mathematical Tables, v. 1)]. Moscow and Leningrad, Gostekhizdat, 1949, 430 p. 14.6 × 22 cm. Boards, 15.70 roubles. An edition of 10000 copies published in March.

This volume contains the following seven tables, several of which are new:

Table 1, p. 8–345: $J_0(x)$, $J_1(x)$ for $x = [0(.001)25; 7D]$, Δ^2.
Table 2, p. 349: 8D values of zeros α_k of $J_0(x)$, β_k of $J_1(x)$, γ_k [except 7D for $k = 1(1)10$] of $J_1'(x)$ for $k = 0(1)40$.
Tables 3–6, p. 350–429: $J_0(\alpha_k x)$, $J_0(\beta_k x)$, $J_1(\beta_k x)$, $J_1(\gamma_k x)$ for $x = [.01(.01)1; 7D]$, $k = 0(1)40$.
Table 7, p. 430: $2[J_1^2(\alpha_k)]^{-1}$, $2[J_0^2(\beta_k)]^{-1} = 2[J_2^2(\beta_k)]^{-1}$, $2\gamma_k^2(\gamma_k^2 - 1)^{-1} \times [J_1^2(\gamma_k)]^{-1}$, for $k = 1(1)40$; 7S].

These last four tables are new and are for calculating terms in the development of a function in a Fourier-Bessel series. From data in the volume it is easy to verify the accuracy of the second and third of these functions, but in the case of the first it is not nearly so easy since the values of $J_1(\alpha_k)$ are not here given—but are, of course, readily available elsewhere.

Most of the values of γ_k are also new; but all the other values of the functions are implied in the Harvard and BAAS tables. The typography of the volume is very unattractive, the paper poor, and the proofreading bad. For example β_{40} is given as 126.1461387, instead of 126.4461387, on pages 388, 389, 408, 409; and there are other errors on pages 6, 7, 340, 347, 348, 374.

R. C. ARCHIBALD

855[L].—J. P. STANLEY & M. V. WILKES, *Table of the Reciprocal of the Gamma Function for Complex Argument*. Computation Centre, University of Toronto, 1950. i + 100 p., 35.4 × 25.1 cm. Price $4.50.

The table is that briefly described in UMT 102 (*MTAC*, v. 4, p. 162). $1/\Gamma(x + iy)$ was computed for $x = -.5(.01).5$, $y = 0(.01)1$ from the infinite series in powers of $x + iy$: twenty-one terms of the series were used. The computations were carried out on the EDSAC in the Mathematical Laboratory at Cambridge University, and checked by differencing in both

directions on a National accounting machine. In order to minimize rounding-off errors, ten decimals were carried throughout the process. In the present volume values are given to 6D, and the authors estimate that the maximum error does not exceed .7 units of the sixth decimal place.

Values of the reciprocal gamma function outside the range of tabulation can be obtained by one or the other of the functional equations satisfied by the gamma function.

The preface (1 p.) gives the following references to available numerical values of the gamma function in the complex domain:

J. G. BECKERLEY, *Indian Jn. Physics*, v. 15, 1941, p. 229–232 [RMT **195**, *MTAC*, v. 1, p. 419–420].
H. T. DAVIS, *Tables of the Higher Mathematical Functions*, 1933, p. 269f.
A. GHIZETTI, Accad. Naz. Lincei, *Atti Rend.*, s. 8, v. 3(2), 1947, p. 254–257 [RMT **617**, *MTAC*, v. 3, p. 415–416].
M. E. LONG, Radar Research Development Establishment, *Memorandum* no. 96, 1945 [RMT **234**, *MTAC*, v. 2, p. 19].
W. MEISSNER, *Deutsche Mathematik*, v. 4, 1939, p. 537–555 [RMT **140**, *MTAC*, v. 1, p. 177].
C. P. WELLS & R. D. SPENCE, *Jn. Math. Phys.*, M.I.T., v. 24, 1945, p. 51–64 [RMT **228**, *MTAC*, v. 1, p. 446].

The preface states that the present table has been checked against all the available values, but it does not state whether any discrepancies were found.

It should be noted that in connection with the tabulation of certain parabolic cylinder functions, a table of $\ln \Gamma(x + iy)$ for $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ is being prepared at the Scientific Computing Service, London, by J. C. P. MILLER [*MTAC*, v. 2, p. 62–63, 147–148, v. 4, p. 90].

A. E.

856[R].—New Zealand Department of Lands and Survey, *Geodetic and Transverse Mercator Projection Tables. Latitudes* 34° *to* 48°. (*International Spheroid*), Wellington, 1946, 96 p. 22 × 33.3 cm.

These are tables of special functions for the Transverse Mercator projection for an area included between latitudes 34° and 48° and longitude 3° each side of a chosen central meridian.

The formula for the x-coordinate in the Transverse Mercator projection may be written

$$x = m + G(\Delta\lambda)^2 + I(\Delta\lambda)^4$$

where m is the meridional distance from the equator to latitude ϕ. G and I are functions of ϕ and of the constants of the meridian ellipse. These functions are tabulated as well as analogous functions for the Transverse Mercator y-coordinate, and for the various inverse formulas expressing latitude and longitude in terms of x and y, etc.

Notable points concerning these tables are:

1. The meridian distance, though tabulated only from 34° to 48°, is given for one minute intervals accurately to 1/1000 link [1 link = 7.92 in.].

2. Detailed examples are given of the geodetic computations involved with the formulas and of the interpolation procedure for the tables.
3. All quantities are in multiples or submultiples of links which would make it necessary to use conversion factors for application to areas where triangulation distances are in meters or feet.

The only detected tabular errors occur on p. 21 and p. 31 and are noted in the volume.

P. D. THOMAS

U. S. Coast and Geodetic Survey
Washington, D. C.

MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **854** (Lîûsternik, Akushskiĭ & Ditkin).

179.—G. F. BECKER & C. E. VAN ORSTRAND, *Smithsonian Mathematical Tables, Hyperbolic Functions*, Washington, fifth reprint, 1942 [*MTAC*, v. 1, p. 45].

On p. 314, in the table of the anti-gudermannian, the value of 43°3′,

for 2667.20 *read* 2867.20

CHARLES T. JOHNSON

5852 Adelaide Ave.
San Diego, Calif.

180.—A. M. LEGENDRE, *Traité des Fonctions Elliptiques*, v. 2, Paris, 1826. In Chapter 3, p. 56 and 58, corresponding to $n = 4$, the coefficient of $\delta^4 f_0$

for $421/(4725 \cdot 2^{10})$ *read* $1/3024$

On p. 58, corresponding to $n = 5$ and $n = 6$ the coefficients of $\delta^4 f_0$,

for $-5/384$ *read* $1/384$
for $-23/1440$ *read* $1/120$

H. E. SALZER

NBSCL

UNPUBLISHED MATHEMATICAL TABLES

110[E].—RICHARD R. KENYON, *Table of* $x^n e^{-x}$. 3 leaves and a graph deposited in the UMT FILE. Photostat.

This is a table of $x^n e^{-x}$ to 5S or 6S for $n = 0(1)8$ and $x = 0(.01).1(.1)5(1)30(5)60$. A graph is included with the tables to show the behavior of the function. It allows rough graphical interpolation to be made for non-integral values of n.

111[F].—A. S. ANEMA, *Primitive pythagorean triangles with their generators and with their perimeters, up to* 119 992, *dedicated to* R. E. POWERS. Typewritten manuscript of 151 leaves, deposited in the UMT FILE.

This remarkable table lists 8431 primitive right triangles arranged according to increasing perimeters. Besides the three sides and the perimeter, the two "generators" of each triangle are given. This table is the basis of the author's UMT **106** [*MTAC*, v. 4, p. 224]. The table contains 350 pairs of triangles having equal perimeters and 6 cases of 3 isoperimetric triangles. The only previous table of this sort is due to KRISHNASWAMI[1] and extends to perimeters $\leq$ 10 000.

[1] A. A. KRISHNASWAMI, "On isoperimetric pythagorean triangles," *Tôhoku Math. Jn.*, v. 27, 1927, p. 332–348.

112[F].—A. GLODEN, *Table de factorisation des nombres* $N^4 + 1$ *dans l'intervalle* $6000 < N \leq 10000$. Manuscript of 70 leaves deposited in UMT FILE.

This is an extension of UMT **108** [*MTAC*, v. 4, p. 224]. In spite of the large size of the numbers $N^4 + 1$ in the range indicated in the title, a good majority of the entries are completely factored. As before, the unknown factors lie beyond 600 000.

113[F].—A. GLODEN, *Table de décomposition en* $a^2 + b^2$ *et* $2c^2 + d^2$ *des nombres premiers de la forme* $8k + 1$ *de l'intervalle* 200 000–300 000 *et Table des solutions des congruences* $l^2 + 1 \equiv 0 \pmod{p}$ *et* $2m^2 + 1 \equiv 0 \pmod{p}$ *pour les mêmes nombres premiers*. Typewritten manuscript of 34 leaves deposited in UMT FILE.

The contents of this table are sufficiently well indicated in the title. The table is a byproduct of the author's table of the solutions of the congruence

$$x^4 + 1 \equiv 0 \pmod{p}$$

[*MTAC*, v. 2, p. 71–72]. The solutions (l, m) are the least positive ones.

114[F].—D. H. LEHMER, *Table of the sum of fifth powers of the divisors of* n *for* $n = 1(1)5000$. Tabulated manuscript and punched cards deposited in UMT FILE.

The function $\sigma_5(n) = \sum_{\delta/n} \delta^5$ occurs as the FOURIER coefficient in the expansion of WEIERSTRASS' invariant g_3. This table of $\sigma_5(n)$ is intended as an aid to research on RAMANUJAN's function $\tau(n)$, [see UMT **101**, *MTAC*, v. 4, p. 162] and was produced with an IBM 602A calculating punch.

115[F].—R. A. LIENARD, *List of primes of the form* $k \cdot 10^6 + 1$ *for* $k < 12000$. Typewritten manuscript, 4 leaves, deposited in the UMT FILE.

This is an extension of UMT **90** [*MTAC*, v. 4, p. 101], which gave 117 values of $k < 1000$, which generated prime numbers. The present table lists 1321 values of $k < 12000$ for which $k \cdot 10^6 + 1$ is a prime number. The author also announces that he has in his possession a manuscript containing complete factorizations of all numbers of the above form for $k < 12000$.

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 225 Far West Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

The BARK, A Swedish General Purpose Relay Computer

Introduction.—In December 1948 The Swedish Board for Computing Machinery decided to build a relay digital computer for which plans and cost estimates had been drawn up by DR. CONNY PALM. Design work started almost immediately under the direction of Dr. Palm. The machine was completed in February 1950 and, after a testing period, was inaugurated on April 28, 1950. The machine is called BARK, standing for "Binär Automatisk Relä Kalkylator." The main characteristics of the computer are listed below.

Numbers are represented by absolute value and sign in the form $2^p \cdot q$, where p is a 6-digit binary number with algebraic sign and q is a 24-digit binary fraction with algebraic sign. Thus, 32 binary characters are required for the representation of one number. In the decimal system this corresponds to a range between 10^{-19} and 10^{+19} with a precision of slightly better than seven decimal digits.

In the number representation, q plus the sign is referred to as the "numerical part" of the number, while the remaining places are called the "exponential part." Similarly, in the following, the 25 places of a register which store the digits and sign of the fraction q will be referred to as the "numerical part" of the register, while the remaining places will be called the "exponential part." Within each part we use the terms "first" or "left(most)" to designate the most significant positions or digits, while "last" or "right(most)" describe the least significant positions.

Storage and Arithmetic Circuits.—The storage consists of 50 relay registers and 100 constant registers, each storing 32 binary characters. In the near future the memory capacity will be increased to 100 relay registers and 200 constant registers. The original design enables this enlargement to be made without difficulty. The constant registers are set manually by means of one 4-position switch for the signs and ten 8-position switches, each one taking care of three binary digits; conversion to binary form is thus necessary before a number is put in a constant register.

Within the machine numbers are transferred along three number transfer busses, each of which contains 32 wires or digit channels. Voltage on a digit channel represents the digit one or a plus-sign; no voltage represents the digit zero or a minus-sign. The A- and the B-busses transmit numbers from the storage to the arithmetic circuits, and the C-bus transmits numbers in the opposite direction.

The arithmetic circuits carry out transfer, addition, and multiplication with arbitrary signs for the numbers involved.

Transfers include transfer with opposite sign, transfer of absolute value and negative absolute value, and logical addition (i.e., the transfer of the logical sum of two numbers). Technically a transfer is accomplished when corresponding digit channels in each of the three busses are interconnected. From this it follows that the ordinary transfer is a special case of logical addition, where one of the two numbers consists of zeros (or minus signs) on all wires.

The adder shifts the number having the smaller exponent to its proper position and then forms the arithmetically correct sum or difference of the two numbers. The shifted number is not rounded off. If the sum of the numerical parts of the two numbers should exceed one, the result is shifted one position to the right and its exponent increased by one. In this case the

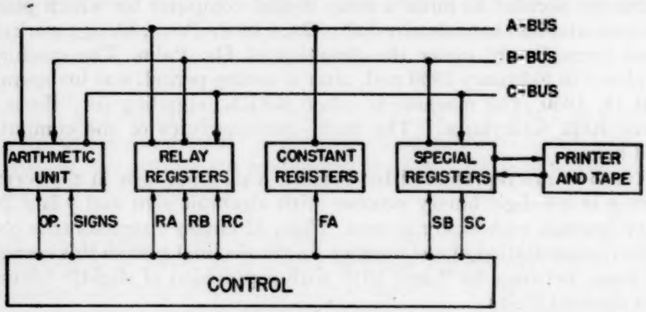


Fig. 1.

result is rounded off before the shift is made. If, on the other hand, the sum or difference should be less than one-half, the result is not shifted. The first digit of the result is therefore not necessarily a one.

The adder contains 232 relays. Addition is carried out in five successive steps. First, exponents and signs are compared. Second, the difference is taken between the two exponents and simultaneously the number to be shifted is selected. Third, the shift is carried out and the number not shifted is inverted, if necessary. (Nines complements are used.) Fourth, the addition of the two numerical parts is carried out. Fifth, the sum is shifted one position to the right, or inverted, or left unchanged, whichever the case may be. Simultaneously the exponent of the result is increased by one if the sum was shifted, and the sign of the numerical part is formed. A chain of slower relays controls the timing, so that no step is started before the previous one is completed.

Multiplication is carried out in the following way: The two numbers coming in are first "normalized," that is, shifted to the left until the first binary one occupies the leftmost position. [This operation is called a "zero-shift" in the report on the BTL Relay Computers (*MTAC*, v. 3, p. 1–13; 69–84).] The exponents are modified accordingly. In the rest of the multiplication the numerical parts of multiplier (MR) and multiplicand (MD) are treated as 8-digit numbers in the octal number system, since a group of three binary digits forms one octal digit. Multiples of MD with the integers 1, 2, $\cdots$, 7 are formed and selected simultaneously by the eight octal digits

of MR. The multiples are added together in seven adders in such a way that only 30 binary digits, or ten octal digits, are retained for the final result. In order to compensate for the digits that are dropped an octal three is added in the tenth octal position. When the whole product is formed, a binary one is added into the twenty-fifth binary position and the 24 first digits retained. The multiplier thus delivers a correctly rounded 24-digit product, where, however, the first digit may be a zero. No provision has been made for computation with double accuracy. Such computation can still be done by a coded sequence of instructions but turns out to be extremely cumbersome and is therefore in most cases impracticable. Altogether the multiplier contains 923 relays.

Normalization of a single number can also be carried out. One of the two normalizing circuits in the multiplier is then connected to the A- and the C-bus. The rest of the multiplier is not affected by this operation.

Other elementary operations where only one number is involved at a time are handled by special circuits. These operations are:

1. Transfer of an exponent to the last six places of the numerical part of a register
2. Transfer of the last six digits of a numerical part into the exponential part of a register
3. Transfer of the numerical part only
4. Transfer of the fractional part of a number (for example, in 23.67 the fractional part is 0.67)
5. Shift one step to the left (or to the right) of the numerical part without modification of the exponent.

Integral numbers are normally represented with the exponent +24, whereby the rightmost digit of the number falls into the last place of the numerical part of a register.

No built-in operation is needed to extract the integral part of a number (53 is the integral part of 53.67), as this may be done by adding to the number a zero with the exponent +24 (an "integral zero"); for in an addition the number having the smaller exponent is shifted to the right until both exponents are equal to +24, and a number with the exponent 24 is an integral number.

Sequencing and Coding.—Instructions for the machine are written in the form: *n* A op. signs B C D, where A and B are the addresses of the two numbers that are combined by the operation "op," which may be T (transfer), A (addition), M (multiplication), or E (special operations on one number only). The "signs" symbol indicates which one of the four possible combinations of signs should be attached to the numbers A and B. The different types of transfers are also distinguished by means of the "signs" symbol, whereas the different E-operations are distinguished by means of the B-address. The letter C is the address where the result should be sent, while *n* is the number, or index, of this particular instruction and D is the number of the next instruction.

Instructions of this type are executed automatically by means of the control system, which has three parts: the central control unit (built into the control table), the "order chain," and the five "order panels." The order

chain contains 840 relays (later it will contain 1200) connected in such a way that one and only one relay can be operated at a time. In order to start the machine on a problem, one of these relays is operated from the control table, and the machine then proceeds to operate automatically. The central control unit now supplies voltages on five lines which pass through contacts on the operated relay of the order chain to plug holes on four of the order panels. On the order panels plugged connections are made so that the voltages reach their destinations. Three of them, coming from the A-, B-, and C-panels, operate relays in the proper registers to connect these registers to the respective busses. The two remaining voltages from the operations panel give the sign-combination and initiate the operation. When the arithmetic unit gives a back-signal to the central control unit indicating that the operation is finished, the five voltages are removed, the relay in the order chain is released, and the next relay is operated. Then the whole procedure is repeated automatically.

Normally the central control unit steps the order chain from relay n to relay $n + 1$. It is possible, however, to break this sequence by means of plugged connections on the sequence panel. This is called a jump in the program. Jumps can be made from any even instruction to any odd instruction. With this restriction only, the whole of the 840 available instructions may be divided into an arbitrary number of subsequences of arbitrary length. These may constitute one or several independent computing programs. Conditional jumps are obtained by means of the selectors which will be discussed below.

The even-odd rule makes it sometimes necessary to insert meaningless instructions in the program. For instance, at the end of a subsequence that would otherwise contain an odd number of instructions, a special phony instruction is available to permit the jump.

No instructions are given to the machine from tapes or similar devices; all programs are physically realized by the plugged connections. For every instruction, normally one such connection must be made on each panel. The principle of the arrangement is similar to the subsequence mechanisms used with Mark I at Harvard and to the corresponding equipment in the BTL computer, Model VI.

Flexibility in the coding is obtained by the use of "selectors" or relay pyramid circuits. There are four 64-selectors (pyramids with 64 exits) and 125 two-way selectors. By letting the path of the corresponding plugged connection go through such a pyramid, any part of an instruction may be subject to a choice of up to 64 different possibilities. The choice is then dependent on some control number previously sent from the arithmetic circuits to the controlling relays of the pyramid. This technique greatly reduces the number of instructions needed in a program. For instance, the multiplication of two square matrices of unspecified order n (where $n < 50$) may be programmed with only 18 instructions.

Other Equipment.—Input and output devices make use of standard teletype equipment. Five tape readers, five tape punches, and one page printer are available. This equipment may also be used as external storage, although with limited flexibility, as no provisions have been made for "hunting" on the tapes. Numbers may be read or punched in binary or decimal notation and printed in octal or decimal notation. Printing in octal

form is used for checking, especially for checking of the setting of the constant registers. Conversion from binary to decimal notation, or inversely, is done by the computer itself and is programmed by ordinary means.

The operation of the machine is supervised from the control table. For checking purposes the order chain may be stepped manually, one instruction at a time, as slowly as desired. Indicator lamps then show at every instant the instruction just carried out, the addresses and operation involved, and the numbers which at that moment occur on the number transfer busses. It is also possible to step the machine manually through arbitrary instructions, not set up on the panels.

No circuits for automatic checking have been built into the machine, but alarms occur for a number of fundamental failures, such as a number exceeding the range of the machine, a dangerous drop in supply voltage, a blown fuse, and, of course, the failure of a coded check.

The machine contains in all some 5000 relays of standard telephone type. The relays are mounted in boxes which can easily be replaced by duplicate units in the case of a breakdown.

A general view of the BARK is shown in the frontispiece. In front is seen the control table and the input and output equipment. At the right and in the first row are the order panels. In succeeding rows are shown the order chain, arithmetic circuits, relay registers, and constant registers. The three empty relay racks provide space for future additions to the machine.

A block diagram of the BARK is shown in Figure 1. Thin vertical lines represent the paths of the signals that initiate operations and effect the connection of the registers to the busses. RA, for example, indicates the path of a signal coming from the A-panel, which connects one of the 50 relay registers to the A-bus.

Operation Times.—The times for elementary operations are:

Transfer	100 milliseconds
Addition	150 milliseconds
Multiplication	250 milliseconds
Printing of one digit	160 milliseconds

The efficiency under realistic conditions may probably be better judged by the following short data from some of the computations which were done during the testing period.

1. Tabulation of four 7th-degree polynomials in two variables (origin: atomic physics). Some 500 values were computed; the machine time was three hours.

2. Tabulation of the specific volume v of water-vapor as a function of the pressure p and temperature $T = 100t$ according to the formula:

$$v = RTp^{-1} - At^{-2.82} - p^2(Bt^{-14} - Ct^{-31.6})$$

R, A, B and C are constants. Some 9000 values were computed: the machine time was 70 hours.

3. Tabulation of irregular solutions of the equation

$$y'' + \left(1 - \frac{2a}{x}\right)y = 0$$

for different values of the parameter a, and $0.05 \leq x \leq 1$ (origin: atomic physics). The equation was solved with step-wise integration, using TAYLOR's series up to and including terms of 7th order, and the result was accurate within one or two units in the 6th decimal place for most values of a. The functions y and y' were computed for about 3000 points. The machine time was 42 hours. The table will be published by C-E. FRÖBERG in *Arkiv för Fysik.*

4. Solution of symmetrical systems of linear equations with the GAUSS' elimination method (origin: surveying). Matrices of orders $n = 8$, 14, 20, and 28 were treated. The machine time for $n = 28$ was 4.8 hours.

5. Inversion of symmetrical and unsymmetrical matrices of orders $n = 8$ and 20 with JORDAN's method (origin: surveying). The machine time for $n = 20$ was 7.5 hours.

Conclusion.—It was found that the coding on BARK is on the whole straightforward and easy. Flow-diagrams, of the type described in the reports on the computing machine under development at the Institute for Advanced Study, Princeton, N. J., are excellently suited for the planning of programs. The plugging of the order panels is time-consuming (the approximate speed being 50 instructions an hour) but may on the other hand be done while the machine is working on some other problem. The greatest advantage of the sequencing system is its flexibility—at any point in a computation the machine may be stopped and the program modified, e.g., by the insertion of a new instruction or subsequence or by skipping another.

The cost of the machine, including planning, designing, construction, and experimental work, does not exceed 100,000 dollars. The main bulk of the design work was done by HARRY FREESE and GÖSTA NEOVIUS. The machine was built by the Swedish Telegraph Administration, which also supplied most of the parts. Under the direction of Dr. Palm, the following persons participated in the general planning, design, and experimental work: C-E. FRÖBERG, O. KARLQVIST, G. KJELLBERG, B. LIND, A. LINDBERGER, P. PETERSSON, and M. WALLMARK.

GÖRAN KJELLBERG
GÖSTA NEOVIUS

Matematikmaskinnämndens Arbetsgrupp
Drottninggatan 95 A
Stockholm, Sweden

DISCUSSIONS

Notes on Numerical Analysis—3
Solution of Differential Equations by Recurrence Relations

1. Introduction.—J. TODD[1] has recently drawn attention to the use of recurrence relations for the numerical solution of differential equations. The characteristic feature of such processes is that successive values of the required solution are computed from earlier values by a recurrence relation involving no estimation. In his note Todd discussed the building-up errors that occur with some of these methods. Though the particular example of catastrophic building-up error given by him is in no way a general criticism of recurrence methods (since the formula in question would be unlikely to be used in actual computation), it is nevertheless true that the problem of building-up error is a serious one with these, as with all methods of solution.

This problem will be dealt with in a later note: the object of the present note is to present some observations on techniques which have arisen in the course of applications of recurrence methods to a variety of equations.

The notes below all concern the application of the "difference-correction" methods described by L. Fox & E. T. Goodwin.[2] These methods have been found to be efficient ones to employ when applicable, from the point of view both of the amount of labor involved and of the smallness of the associated building-up error. For convenience only second-order equations are treated but similar considerations hold for equations of other orders.

2. The η-process.—Consider the solution of the equation

$$y'' + f(x)y = 0, \tag{1}$$

for a chosen interval of integration h. For convenience we take as the initial conditions,

$$y(x_0) = a, \qquad y(x_0 + h) = b, \tag{2}$$

reserving the more usual case until section 4. Method VII of Fox & Goodwin is based on the recurrence relation

$$(1 + \phi_1)y_1 = (2 - 10\phi_0)y_0 - (1 + \phi_{-1})y_{-1} + \Delta y_0, \tag{3}$$

where $\phi(x) = \frac{1}{12}h^2f(x)$, Δy is given by

$$\Delta y = \left(-\frac{1}{240}\delta^6 + \frac{13}{15120}\delta^8 - \frac{17}{100800}\delta^{10} + \frac{1}{29700}\delta^{12} - \cdots\right)y, \tag{4}$$

and the subscripts -1, 0, 1 applied to y and ϕ are taken to represent any three successive points at the interval h.

This recurrence relation is solved by an iterative process. A first approximation $y^{(1)}$ is obtained by neglecting the difference correction. The latter is then estimated from the differences of $y^{(1)}$ and used in (3) to obtain a second approximation $y^{(2)}$. This process is continued until no further changes occur.

A variation of this technique is the calculation of successive corrections rather than successive approximations, and it results in an appreciable saving of labor when desk calculating machines are being employed and many figures retained.

The second approximation $y^{(2)}$ is written in the form $y^{(2)} = y^{(1)} + \eta$, where η is the first correction and satisfies the recurrence relation

$$(1 + \phi_1)\eta_1 = (2 - 10\phi_0)\eta_0 - (1 + \phi_{-1})\eta_{-1} + \Delta y_0^{(1)}, \tag{5}$$

and the initial conditions $\eta(x_0) = \eta(x_0 + h) = 0$. The η may be calculated from these relations, and, since it will contain many fewer figures than $y^{(2)}$, its computation is correspondingly more rapid. This process can now be repeated to produce a second correction $\eta^{(2)}$ which will contain even fewer figures than η, and so on.

An apparent drawback of this procedure is that the probable building-up error is increased since y is now obtained as a sum of the first approximation $y^{(1)}$ and successive corrections η, each of which will contain its own rounding errors. However these errors are at most additive, and, even if several successive corrections are obtained, the resulting accumulation of rounding-off error will be more than compensated by the retention of one extra guarding

figure. In the more usual case, when only the first correction is significant, even this safeguard will usually be unnecessary.

This "η-process" is of general application to the solution of linear differential equations by difference-correction methods, and in the case of simultaneous linear equations, simultaneous corrections can be obtained in a similar way.

3. Non-linear Equations.—Though the η-process is useful when employed in this way it is of much greater value in the solution of non-linear differential equations. Consider the differential equation

$$y'' + f(x, y) = 0. \tag{6}$$

This equation can be solved very readily by Method VI of Fox & Goodwin's paper, which leads to the recurrence relation

$$y_1 = 2y_0 - 12\phi(x_0, y_0) - y_{-1} + \Delta y_0, \tag{7}$$

where $\phi(x, y) = \frac{1}{12}h^2f(x, y)$ and Δy is given by

$$\Delta y = \left(\frac{1}{12}\delta^4 - \frac{1}{90}\delta^6 + \cdots\right)y. \tag{8}$$

Method VII is much more powerful, however, in the sense that it involves a very much smaller difference correction. The recurrence relation is

$$y_1 + \phi(x_1, y_1) = 2y_0 - 10\phi(x_0, y_0) - y_{-1} - \phi(x_{-1}, y_{-1}) + \Delta y_0, \tag{9}$$

where Δy is given by (4).

A direct application of equation (9) involves the solution at each step of a non-linear equation for y_1, usually best performed by a method of successive approximation such as Newton's rule, and this rather awkward process has to be repeated for each approximation. By employing the η-process, however, it is usually possible to determine the correction η to an approximate solution $y^{(1)}$ from a *linear* recurrence relation as follows.

The first approximate solution $y^{(1)}$ is obtained by using (9) replacing Δy_0 by zero. Writing

$$K_1 = 2y_0^{(1)} - 10\phi(x_0, y_0^{(1)}) - y_{-1}^{(1)} - \phi(x_{-1}, y_{-1}^{(1)}), \tag{10}$$

the equation to be solved for $y_1^{(1)}$ at each step is

$$y_1^{(1)} + \phi(x_1, y_1^{(1)}) = K_1. \tag{11}$$

If $a^{(1)}$, $\phi(x_1, a^{(1)})$ are taken as approximations to $y_1^{(1)}$, $\phi(x_1, y_1^{(1)})$ respectively, then the next approximations are

$$a^{(1)} + \delta a,\ \phi(x_1, a^{(1)}) + \delta a \phi_y'(x_1, a^{(1)}), \tag{12}$$

where

$$\delta a = [K_1 - a^{(1)} - \phi(x_1, a^{(1)})]/[1 + \phi_y'(x_1, a^{(1)})]. \tag{13}$$

If $a^{(1)}$ is correct to k figures the new approximations (12) will generally be correct to $2k$ figures. A good first approximation $a^{(1)}$ can be extrapolated from the differences of $y^{(1)}$, so that in practice only one application of (13) is necessary.

To obtain the second approximate solution $y^{(2)}$, write $y^{(2)} = y^{(1)} + \eta$ as before. Then η satisfies the recurrence relation

$$(14) \quad [1 + \phi_y'(x_1, y_1^{(1)})]\eta_1 = [2 - 10\phi_y'(x_0, y_0^{(1)})]\eta_0 - [1 + \phi_y'(x_{-1}, y_{-1}^{(1)})]\eta_{-1} + P + \Delta y_0^{(1)},$$

where P comprises terms involving second and higher powers of η. The quantity $\phi_y'(x_1, y^{(1)})$ will already have been tabulated to a sufficient accuracy during the calculation of $y^{(1)}$. Thus, if P is neglected, η can be rapidly computed from a linear recurrence relation. The value of P can subsequently be estimated, and, if necessary, its effect, and also that of $\Delta\eta$, can be allowed for by a further correction $\eta^{(2)}$, though in all the examples on which we have employed this method P and $\Delta\eta$ have both turned out to be negligible. When subsequent reference is made to (14), it will be assumed that P is negligible.

4. Initial Conditions Involving the Derivative.—In the above discussion it has been supposed that two consecutive initial values of the required solution have been available. Frequently, however, the initial conditions may involve the derivative, taking the form

$$(15) \qquad y(x_0) = a, \qquad hy'(x_0) = c.$$

For a linear differential equation this case usually presents little difficulty since $y(x_0 + h)$ can be obtained from the Taylor series at x_0, the successive derivatives being obtained by repeated differentiations of the equation. With a non-linear equation, however, this process may be a tedious one, due to the difficulty of obtaining the higher derivatives. Though it is always possible to use a very small interval it is generally preferable to use an extension of the "η-process" by means of which most of the higher derivatives can be avoided entirely without reduction of the interval.

Suppose for example it is desired to carry out the integration to ten decimals in y and that throughout the range of integration it is known that $|\phi_{yy}''(x, y)| < 0.1$. In these circumstances the Taylor series is used to determine $y(x_0 + h)$ accurately to *five* decimals only, and, if the value so obtained is denoted by β, then starting with the initial values

$$y^{(1)}(x_0) = a, \qquad y^{(1)}(x_0 + h) = \beta,$$

a first approximate solution $y^{(1)}$ is obtained to ten decimals from the recurrence relation (9), replacing Δy_0 by zero. The solution $y^{(1)}$ is extended for two or three backward steps of the argument from x_0 and the value of $h(dy^{(1)}/dx)$ at $x = x_0$ is determined by central numerical differentiation. Clearly the required solution of the differential equation will be $y^{(1)} + \eta$ where η satisfies (14) and has the initial values

$$(16) \qquad \eta(x_0) = 0, \qquad h\eta'(x_0) = c - h(dy^{(1)}/dx)_{x=x_0} = d.$$

Thus the original problem of starting the solution of the equation for y with the conditions (15) is now replaced by an exactly similar problem for η with the conditions (16). The advantage of transforming the problem into this form lies of course in the fact that the recurrence relation satisfied by η is a linear one.

The actual starting of the η-integrations can be carried out as follows. Let $\eta^{(1)}$ be any particular solution of (14) such that $\eta^{(1)}(x_0) = 0$ and let η^* be any solution of

$$(17)\quad [1 + \phi_y'(x_1, y_1^{(1)})]\eta_1 = [2 - 10\phi_y'(x_0, y_0^{(1)})]\eta_0 - [1 + \phi_y'(x_{-1}, y_{-1}^{(1)})]\eta_{-1},$$

such that $\eta^*(x_0) = 0$, $\eta^*(x_0 + h) \neq 0$. Then all the solutions of (14) which vanish at $x = x_0$ can be written in the form $\eta = \eta^{(1)} + A\eta^*$, where A is an arbitrary constant. The solution satisfying (16) has A given by $A = (d - d_1)/d^*$, where $d_1 = h(d\eta^{(1)}/dx)_{x=x_0}$, $d^* = h(d\eta^*/dx)_{x=x_0}$, and can be determined by numerical differentiation. Thus the initial values of the desired η can be obtained by constructing numerically two solutions $\eta^{(1)}$, η^* of (14) and (17) respectively in the neighbourhood of $x = x_0$ and forming their appropriate linear combination. For convenience $\eta^{(1)}$ is chosen to be as near the true η as possible, for example the value of $\eta^{(1)}(x_0 + h)$ could be taken as d.

5. Examples.—Non-linear differential equations of the type (6) that have recently been solved at the Mathematics Division by application of difference-correction methods have been the differential equation for the modulus of the Hankel function $H_n^{(1)}(x)$ and Emden's equation. The former of these is required incidentally in the computation of the early zeros of the Bessel functions $J_n(x)$, $Y_n(x)$; its application in this connection is described fully elsewhere.[3] The actual equation which is solved numerically is

$$(18)\qquad \frac{d^2u}{dx^2} = \frac{\pi^2}{u^3} - \frac{x^2 - n^2 + \frac{1}{4}}{x^2}\,u,$$

the desired analytical solution being

$$u = 2^{-\frac{1}{2}}\pi x^{\frac{1}{2}}[J_n^2(x) + Y_n^2(x)]^{\frac{1}{2}}.$$

Integrations of (18) were carried out to eleven decimals in u over the range $n \leqslant x \leqslant n + 5\pi$, the interval in x being $\frac{1}{4}$ or $\frac{1}{2}$. One run with the recurrence formula (9) with Δy_0 replaced by zero, followed by a single application of the "η-process," produced the required solution u correct to eleven decimals. Initial values $u(n)$, $u'(n)$ were obtained from asymptotic expansions and the integrations were in effect started by the device described in the previous section. An analysis of the recurrence relations indicated the building-up error in u to be of an oscillatory nature and of magnitude not exceeding about three or four units in the eleventh decimal. This was verified in one or two instances by repeating the solution retaining a different number of decimals.

Emden's equation can be written in the form

$$\frac{d^2z}{dx^2} + x\left(\frac{z}{x}\right)^n = 0,$$

and solutions of this for $n = 1\frac{1}{4}(\frac{1}{2})4\frac{3}{4}$ with the initial conditions $z(0) = 0$, $z'(0) = 1$, have been prepared for the Royal Society Tables Committee over the ranges $0 \leqslant x \leqslant x_0$ where x_0 is the first zero of z. The number of steps

involved increases with n and varied from about 30 to 150. Ten decimals were retained in z for the lower values of n and eleven for the higher values. The building-up error here was more severe than that associated with the equation (18), but even so, at least eight decimal accuracy could always be guaranteed in z. Again one run of (9) taking the difference correction as zero, followed by a single application of the "η-process," was adequate to produce the solution z to the requisite accuracy.

This note is published with the permission of the Director of the National Physical Laboratory.

C. W. Clenshaw
F. W. J. Olver

National Physical Laboratory
Teddington, Middlesex
England

[1] J. Todd, "Solution of differential equations by recurrence relations," *MTAC*, v. 4, p. 39–44.
[2] L. Fox & E. T. Goodwin, "Some new methods for the numerical integration of ordinary differential equations," Cambridge Phil. Soc., *Proc.*, v. 45, 1949, p. 373–388.
[3] F. W. J. Olver, "A new method for the evaluation of zeros of Bessel functions and of other solutions of second order differential equations," Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 570–580.

Bibliography Z–XIV

1. Anon., "Computers that correct their own errors," Franklin Institute, *Jn.*, v. 249, 1950, p. 515.

This is a brief note on the development at the Bell Telephone Laboratories of a method for endowing high-speed computers with a self-correcting faculty.

2. Anon., "Electronic computers," I. R. E., *Proc.*, v. 38, Apr. 1950, p. 373–375.

Bibliography of digital and analogue computers.

3. Anon., "New memory tube," *Electronics*, v. 23, May 1950, p. 130.

A new electrostatic storage tube developed at Massachusetts Institute of Technology was announced at the IRE convention in New York last March.

It is a beam-deflection tube which stores binary information at two stable potentials, 100 volts apart. A 100-volt electron flood replaces leakage and maintains the stored information indefinitely. A single 2,000-volt electron beam writes or reads one of 400 binary digits on a four-inch target. The potential boundary is kept stable on the storage surface by a mosaic of conducting beryllium squares. The access time is 25 μ sec. At the present time tubes are in pilot production for use in a digital computer. It is hoped that future developments will decrease access time to 6–12 μ sec. and increase storage capacity to 1,024 binary digits.

M. M. Andrew

NBSMDL

4. ANON., *Swedish Automatic Relay Computer*, July 20, 1950, 4 pages, mimeographed, Office of Naval Research, London, Technical Report ONRL-67-50.

The report presents a brief description of the recently completed Swedish relay machine, Binär Automatisk Relä-Kalkylator or BARK [*MTAC*, v. 4, p. 53, 118, v. 5, p. 29–34].

5. ANON., NBS, *Technical News Bulletin*, v. 34, Sept. 1950, p. 121–129.

Included in this issue is a description of the National Bureau of Standards Eastern Automatic Computer (SEAC) including sections on the solution of the skew-ray problem (SEAC's first application in the physical sciences), on the solution of a heat-flow problem, and on the NBS computer program.

6. M. L. BYKHOVSKIĬ, "Osnovy electronykh matematicheskikh mashin discretnovo scheta" (Principles of electronic mathematical digital computing machines), *Uspekhi Matematicheskikh Nauk*, v. 4, 1949, p. 69–124.

Although this article gives a full account of several machines already constructed or in the process of construction—in the United States and England, no mention is made of these facts, nor is any machine alluded to by name. The article is replete with diagrams, photographs, and charts taken from publications in this country and in England, but no mention is made of their sources. The ENIAC and SSEC designs are treated in great detail, and much space is given to a description of various memory devices, including the Selectron and the Williams memory tube. The only Russian contribution that the reviewer could find was a reference to G. B. VOĬTILLA's textbook entitled, *A General Course in Radiotechnology*, Moscow, 1948.

IDA RHODES

NBSMDL

7. R. W. HAMMING, "Error detecting and error correcting codes," *The Bell System Technical Journal*, v. 29, 1950, p. 147–160.

This paper is divided into two main parts. The first part explains principles for designing error detecting and correcting codes for binary equipment, with emphasis on (a) single error correcting codes and (b) single error correcting and double error detecting codes.

The use of such codes may be illustrated by the single error correcting code. In this kind of code we assume the code symbols to be n binary digits in length, m digits of which are associated with the information while $k = n - m$ digits are used for detection and correction of a single error. Having received a code symbol, we derive from it a k-position "checking number" (to be distinguished from the k error detecting and correcting positions in the code symbol as transmitted) which will be 0 if the number received is correct in all n positions. If the received number is incorrect in a single digit position, the checking number indicates the position. The author also derives criteria for the "redundancy" $R = n/m$ to be a minimum.

The second part of this paper discusses the general theory of error detecting and correcting codes in terms of a geometric model. If we assume a code requiring n binary digits, the model consists in setting up a corre-

spondence between the code symbols and the corresponding vertices of a unit n-dimensional cube. The code points form in general a proper subset of the set of all vertices of the cube. "Distance" between two points in this space of 2^n points is defined as the least number of edges which must be traversed in going from one point to the other. This distance function satisfies the conditions for a metric. Criteria are developed for single error detection, single error correction, single error correction plus double error detection, etc., in terms of minimum "distance" between code points. The previously derived conditions for minimum redundancy are again derived by use of the geometric model.

JOSEPH H. LEVIN

NBSCL

8. JAMES H. KNAPTON & LOUIS D. STEVENS, "Gate-type shifting register," *Electronics*, v. 22, Dec. 1949, p. 186–192.

This shifting register is most notable for its economy of components. Each stage contains only one double triode, eleven resistors, five capacitors, and two germanium diodes connected to form an Eccles-Jordan type flipflop and two simple gate circuits. Shift pulses are applied simultaneously to every grid of every stage, but each is gated by the potential of the corresponding plate of the preceding stage. Thus the necessary memory for the information in transit resides in the plate capacitances, because to gate the shift pulse for the next stage a plate must maintain its former potential throughout the duration of an effective flipping pulse on the grids. The gate circuits are amplitude sensitive in the sense that a shift pulse that is too large can override gating by the plate potential. This register has shifted at 600 kc. and is said certainly to shift reliably up to 250 kc. While shifting registers which are both faster and which deliver more useful output current per stage have been built, the economy of this circuit may, for suitable applications, offset these as well as the disadvantages of amplitude sensitive gating.

R. D. ELBOURN

Electronic Computers Section
NBS

9. MOORE SCHOOL OF ELECTRICAL ENGINEERING, *A Functional Description of the EDVAC.* 2 v., Philadelphia, 1949, University of Pennsylvania, Mimeographed.

This publication is a two-volume report including 20 pages of preface and a table of contents, 212 pages of text, 73 pages of appendices, and 47 drawings. A list of the authors is included on page II of the preface.

The text is divided into nine chapters. These are: (1) Introduction, (2) Dispatcher, (3) The EDVAC Control, (4) Computer, (5) Memory, (6) Reader-Recorder, (7) Timer, (8) Power Supply, and (9) Switchgear.

There are five appendices, which are: (1) Physical Description, (2) List of Remote Connections, (3) Definitions and Abbreviations, (4) General Principles of Crystal-Diode Gating Circuits, and (5) EDVAC Drawings.

The purpose of this report can best be stated by quoting three sentences from page III of the preface. "It is believed that personnel assigned to the maintenance of the EDVAC will find this logical or functional description

of much greater value in their work than a circuit analysis. It is hoped that this report will offer more than an aid to maintenance. It contains a description of the pulse and logical techniques used in the EDVAC and thus will prove of interest to all those in the digital computing and allied fields."

As far as content is concerned, Chapter 1 contains a short description of the various units comprising the EDVAC, as well as a summary of the codes used and the speeds at which various functions are performed. Chapter 2 discusses the circuits used to interpret the coding and to supply operating pulses to the computer, as well as the halt circuitry, the extract circuitry, and the memory systems necessary for the dispatcher. Chapter 3 covers the buttons, lights, and switches by which the operator controls the EDVAC. This chapter also explains both the circuits controlled by the buttons and switches and those controlling the lights. Chapter 4 describes the computing unit, the circuits used in the basic arithmetic operations, the "carry" and "borrow" circuits, and the computer check circuits. Chapter 5 discusses the memory tanks and the circuits used in reading numbers into and out of the memory. Chapter 6 covers the reader-recorder equipment (not completed at the time of publication), the circuits to be used in reading from the magnetic tapes to the EDVAC, and the circuits reading information from the EDVAC to the magnetic tapes. Chapter 7 describes the timer and the circuits used to synchronize the operations of the various units of the EDVAC. Chapter 8 describes the power supply and the voltage regulating circuits. Chapter 9 discusses the switches which control the application of power to the EDVAC and the fuse and thermocouple safety systems.

Appendix 1 describes the physical size and layout of the EDVAC, gives some statistics on the number of tubes, crystals, and other basic components used, and lists the lettering system used to identify any particular pin on any plug of any chassis. Appendix 2 lists all the remote connections—pulses used on any drawing which do not originate on that drawing and the drawings on which they originate. Appendix 3 lists the abbreviations used and defines many of the electronic terms used in the report. Appendix 4 discusses in some detail the theory of crystal gating, particularly for those gates used in the EDVAC. Appendix 5 lists the drawings included in the report and explains the system of numbering drawings.

The report is completely built around the 47 logical drawings included at the end of the text. It should be emphasized that these are not circuit drawings with all the details of tubes and resistors and condensers necessary to construct the machine, but rather logical drawings in terms of "and" circuits, "on" circuits, and "inhibitors" which show how the machine makes its necessary decisions. It is impossible to read intelligently any of the chapters of the main report, except Chapter 1, without constant reference to these drawings. However, by carefully following text and drawings simultaneously, the logic of the circuits is completely explained and not too difficult to follow.

The text itself was written by several authors, each writing one chapter. It suffers in some respects from lack of proper editorial supervision. The early chapters are written using a completely logical notation which ignores all specific circuit elements. The last sentence of Chapter 6 reads, "PAcc F4C4 that energizes Ff F4C3 controlling /WM/ output only searches for a pulse in PP32 by requiring coincidence between /ZC/ and a command

pulse at time 32." The following sentence, taken from page 8–4, is typical of the later chapters: "In the 6J6 plate circuit a resistor network provides a d.c. stepdown to vary the bias of a type 6AC7 pentode d.c. amplifier." There may be a good reason for this shift to specific detail, but it is not apparent. This change was quite confusing, particularly as the diagrams accompanying these sections were detailed circuit diagrams rather than logical diagrams. Also, many of the early chapters were far too long for the amount of explanation contained therein.

This report undoubtedly fulfills its first purpose. The machine is completely described in a manner well suited for maintenance work. It falls short on its other objectives. As far as pulse techniques and logical techniques go, there is little of pulse techniques and a not-too-adequate summary of the logical techniques used in the EDVAC. It will be of interest primarily to persons working in the field of machine design and construction and of little interest to persons using machines to solve problems.

EVERETT C. YOWELL

NBSINA

10. GEORGE W. PATTERSON, *Bibliography of Unclassified Moore School Reports on Electronic Digital Computers*, Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, 1950, 3 p.

11. OFFICE OF NAVAL RESEARCH, *Digital Computer Newsletter*, v. 2, no. 3, August 1950, 3 p.

The present status of the following digital computer projects is treated briefly in this number.

1. The Aberdeen Proving Ground Computers
2. The Institute for Advanced Study Computer
3. National Bureau of Standards Eastern Automatic Computer (SEAC)
4. National Bureau of Standards Western Automatic Computer (SWAC)
5. Project Whirlwind
6. MADDIDA
7. Raytheon Computers
8. The EDSAC, Cambridge University, England

12. AN WANG & WAY DONG WOO, "Static magnetic storage and delay line," *Jn. Appl. Phys.*, v. 21, 1950, p. 49–54, figs.

Several years ago HOWARD AIKEN, Director of the Computation Laboratory at Harvard University, proposed a memory device which does not require a continual supply of power to maintain storage. It is a magnetic core that may be in either of two states of residual induction according to the direction in which it was last saturated. The state of the core may be sensed by pulsing it strongly. If the residual induction was already in the direction of the pulse, the induction will change very little; but, if the residual induction was in the opposite direction, the pulse will reverse the induction, and this large change can induce a pulse in another winding. With a suitable coupling circuit using one or two resistors or selenium rectifiers, the state of one core can be transferred to another by the sensing pulse which thus

becomes a shifting pulse. In this manner magnetic flipflops and shifting registers (delay lines) can be built. Work on these devices under contract W19-122-AC-24 with the U. S. Air Force has been reported regularly in the quarterly progress reports of the Computation Laboratory and is well summarized in this paper.

The speed of reversing the induction in a core is limited by eddy currents; hence the time for reversing varies directly with the lamination thickness and inversely with the impressed voltage. For 0.001 inch thick laminations of 50% nickel-iron alloy and a reasonable impressed voltage the reversing time is 10 microseconds; therefore, the shifting frequency is limited to 30 or 50 kc.

The recently announced commercial availability of magnetic strip 0.00016 inch thick (see *Iron Age*, Aug. 10, 1950, p. 90) may permit some increase over the speed reported in this paper. The factor of six in thinness might be expected to produce an increase in speed by the same factor; however, before this factor is achieved, leakage inductance and stray capacitance will become so significant as to require a more careful analysis of the device's operation.

R. D. Elbourn

Electronic Computers Section
NBS

13. M. V. Wilkes & W. Renwick, "The EDSAC—an Electronic calculating machine," *Jn. Scientific Instruments and of Physics in Industry*, v. 26, 1949, p. 385–391.

News

Association for Computing Machinery.—The fall meeting of the Association was held on Sept. 7 through 9 in Washington, D. C. In addition to the meetings, there was a demonstration of the National Bureau of Standards Eastern Automatic Computer (SEAC) at the National Bureau of Standards Electronics Laboratory.

The program for the meeting was as follows:

September 7, Evening Session	J. H. Curtiss, NBS, *Chairman*
"The impact of high-speed computing on atomic research"	R. D. Huntoon, NBS
September 8, Morning Session	R. F. Clippinger, BRL, *Chairman*
"The federal computing machine program"	Mina Rees, Office of Naval Research
"High-speed computation in programming and planning"	J. von Neumann, Institute for Advanced Study
"The effect of high-speed computing on mathematical research"	D. H. Lehmer, University of California, Berkeley
Afternoon Session (A)	M. V. Hansen, Bureau of the Census, *Chairman*
"Remote control demonstration of the SEAC"	R. J. Slutz, NBS
"Operational experience on the EDSAC"	M. V. Wilkes, University of Cambridge
"A photographic high-speed printer (300–3000 characters per second)"	H. L. Daniels, Engineering Research Associates, Inc.
"Provision for expansion of the SEAC"	A. L. Leiner, NBS

Panel discussion on the potentialities and the need for electronified filing and records systems. Participants:

R. R. SHAW, Department of Agriculture
M. E. MCGEOGHEGAN, Department of Treasury
J. RABINOW, NBS
D. SAVAGE, Remington Rand, Inc.
I. TRAVIS, Burroughs Adding Machine Co.
H. L. DANIELS, Engineering Research Associates, Inc.

Afternoon Session (B)	C. V. L. SMITH, Office of Naval Research, *Chairman*
"Number-theoretical problems on the SEAC"	J. C. P. MILLER, NBS
"On the numerical solution of one-dimensional aerophysical problems involving shocks"	R. J. SEEGER & H. POLACHEK, U. S. Naval Ordnance Laboratory
"Numerical methods in the theory of linear partial differential equations of elliptic type"	S. BERGMAN & M. M. SCHIFFER, Harvard University
"Some recent experiments with the Monte Carlo method"	J. TODD, NBS
"Application of the SEAC to linear programming"	G. B. DANTZIG, Office of the Air Comptroller, USAF
September 9, Morning Session	E. W. CANNON, NBS, *Chairman*
"Operational experience on the EDSAC (Part II)"	M. V. WILKES, University of Cambridge, England
"Applications of high-speed computing in aeronautical research"	H. L. DRYDEN, National Advisory Committee for Aeronautics
"The role of a central computing laboratory"	J. H. CURTISS, NBS
Afternoon Session	E. D. SCHELL, Office of the Air Comptroller, USAF, *Chairman*
"Integrating systems of ordinary differential equations"	L. H. THOMAS, IBM
"Additive constants in machine programs"	M. H. A. NEWMAN, University of Manchester, England
"On the non-iterative numerical solution of boundary value problems"	M. A. HYMAN, U. S. Naval Ordnance Laboratory
"Some thermodynamic tables obtained on punched-card machines"	J. BELZER, The Ohio State University
"The determination of a potential using the Fairchild Linear Equation Solver"	C. L. PERRY, Oak Ridge National Laboratory

Institute for Applied Mathematics of the Swiss Federal Institute of Technology.—In July 1950 a sequence controlled computer was installed at the Institute located in Zurich. The computer, which is similar to the Bell Computer Model 5, was constructed by Konrad Zuse with the collaboration of the Institute. (See *MTAC*, v. 2, p. 355–359 for an earlier description of the Zuse computer.)

This relay computer uses 2200 telephone relays and 20 stepswitches and contains a new type of mechanical storage element developed by Zuse. At present the machine has a storage capacity of 64 numbers with an access time of .5 sec. It is hoped that the capacity will eventually be increased to 1024 numbers.

It is a binary computer with a floating binary point and with the binary exponent ranging in value from +63 to −64. Binary to decimal conversion is fully automatic. The machine has the special faculty of handling values such as 0, ∞, and $\frac{0}{0}$ by the storage and arithmetical unit to prevent machine stoppage due to an overflow.

The computer is essentially a one-bus machine, each order containing an operation or an operand. Input is accomplished by a ten-place keyboard or by punched tape; output is typewritten or punched on tape which can be refed to the tape reading unit thus providing an external storage.

The following orders are performed on the machine: add, subtract, multiply, divide, square root, form absolute value, conditional and unconditional call, conditional and unconditional stop, and conditional skip. (The skip order causes all the following orders up to a starting order to be disregarded and hence permits several subsequences to be punched on one loop of tape.) The total times for addition and subtraction, multiplication, and division and square rooting are approximately .1 sec., 2.5 sec., and 6 sec., respectively. To skip an order requires .2 sec.

A problem preparation unit similar to the coding machine for Mark III has been used in preparing the sequence tape. Unlike the coding machine, the keyboard for this unit can be used to operate the computer manually. The coded sequence is punched into movie film (each order being 8 binary digits in length). This is fed into one of the two reading stations. Provisions are now made for additional reading stations.

Institute for Numerical Analysis.—On August 17, 1950, the National Bureau of Standards Western Automatic Computer (SWAC) was formally dedicated at the Institute, University of California, Los Angeles. The machine was sponsored by the Office of Air Research of the United States Air Force for use by the Institute in long-range mathematical research as well as on present-day problems originating with the Air Force, Air Force contractors, and other governmental agencies. Two primary functions of the Institute are to carry on long-range fundamental research in various fields of mathematics related to the effective use of automatic digital computing machinery and to provide computing services to western scientific laboratories. The research program is financed principally by the Office of Naval Research while the computation unit is financed chiefly by the Office of Air Research. The completion of SWAC greatly improves the computing facilities of the Institute and increases the effectiveness of the research program. (For a description of SWAC, see *MTAC*, v. 4, p. 103–108.) Speakers at the dedication ceremonies were E. U. Condon, NBS; Col. F. S. Seiler, OAR; L. N. Ridenour, Univ. of Ill.; J. H. Curtiss, NBS; and H. D. Huskey, INAMDL. The ceremonies were followed by a demonstration of the SWAC.

The SWAC is an extremely fast (16,000 additions of ten-digit numbers and 2,500 multiplications per second) automatically sequenced electronic digital computer. Cathode ray tubes operating on a principle discovered by F. C. Williams of Manchester University, England, are used as the storage element in the high-speed memory unit. This unit is parallel with (initially) thirty-seven binary digits per word or number. Its access time is 16 μ sec. This type of memory requires regeneration, which is accomplished during alternate eight-microsecond intervals. During the other eight-microsecond intervals, operands are transferred from the memory to the arithmetic unit, results are transferred back to the memory, and the next command is transferred to the control unit.

The arithmetic unit also operates in parallel fashion. The net addition time, about five microseconds, is determined by the time required for carry to take place in respect to all of the thirty-seven digits of the word. The computer operates in a synchronous manner, the full five microseconds being allowed for each addition whether carry takes place or not.

The arithmetic unit performs the operations of addition, subtraction, both rounded off and exact multiplication, normal and absolute comparison (which changes the course of the computation depending on the relative sizes of two numbers), and extract (which divides numbers up into parts which the computer can then handle in different ways). More elaborate operations than these are accomplished by routines of instructions stored in the memory.

The initial input-output equipment consists of electromatic typewriters and punched paper tape units. Routines for solving standard problems will be established on tape and stored in a "library."

An auxiliary memory of a magnetic drum is now being built and will be added to the computer, increasing the computer's total storage capacity to over 8000 words. A magnetic

tape input-output unit is also being added to the computer. It is to be used as a slow-speed memory and will have a capacity of about 180,000 words.

The computer and its auxiliary equipment occupy about 50 square feet of floor space. Only standard components are used throughout the computer. All circuitry is on plug-in units, and there are spare plug-in units for about 80% of the chassis in the computer. This type of construction, together with certain borderline checking facilities, will, it is hoped, mean a small percentage of down-time for the computer.

The research computing on the machine will include such problems as matrix inversion, finding characteristic values of matrices, solution of simultaneous linear equations, finding complex roots of algebraic equations, etc. A problem in pure mathematics for which it is planned to use the machine is the computation of zeros of the Riemann-zeta-function. Computation of more roots would lead to further information on the distribution of primes and might provide the key steps for a proof or disproof of this famous conjecture.

The dedication ceremonies were followed on August 18th by a symposium on the applications of digital computing machinery to scientific problems. The purpose of the symposium was to interchange information on various scientific problems which are now being studied by West Coast laboratories and universities and to which high-speed automatic digital computing machinery may be applicable.

The program for the meeting was as follows:

Introductory Remarks	E. U. Condon, Director, NBS
Morning Session	E. P. Little, Office of Air Research, U.S.A.F., *Chairman*
"Initiation of an airplane turn"	Ellis Lapin, Douglas Aircraft
"Problems in water entry ballistics"	E. P. Cooper, U. S. Naval Ordnance Test Station, Pasadena
"Reduction of measurements in free flight testing of missiles"	Elmer Green, U. S. Naval Ordnance Test Station, Inyokern
"Solution of games by iterative processes"	Paul Armer, RAND Corporation
"Nuclear reactor physics computations"	Sidney H. Browne, North American Aviation, Inc.
"The use of iterative processes in the solution of partial differential equations"	Stanley Frankel, California Institute of Technology
"A problem of the Naval Air Missile Test Center"	L. H. Cherry, U. S. Naval Air Missile Test Center, Point Mugu
Afternoon Session	H. D. Huskey, NBS, *Chairman*
"Some problems in mathematical statistics"	Jerzy Neyman, University of California, Berkeley
"An iterative construction of the optimum sequential decision procedure when the cost function is linear"	Lincoln Moses, Stanford University
"Problems in pure mathematics"	D. H. Lehmer, University of California, Berkeley
"On the Green's function of the clamped plate"	Paul R. Garabedian, Stanford University
"Perturbations of a satellite rocket"	Samuel Herrick, University of California, Los Angeles
"Physics research problems at Stanford susceptible to automatic computation"	Paul H. Kirkpatrick, Stanford University
"An astronomical problem"	Leland E. Cunningham, University of California, Berkeley
"Automatic computation in rocket engine research"	H. L. Coplen, Aerojet Engineering Corp.

Moore School of Electrical Engineering, University of Pennsylvania.—The University is offering a graduate course for 1950–1951 in electrical engineering with emphasis on large-scale computing devices. The course of study includes the following subjects: transient circuit analysis, engineering physics, electronics, introduction in digital computing machines, engineering techniques for solving differential equations, servomechanisms and feedback control, continuous variable computers, digital computers-logic, digital computers-engineering principles, advanced topics in numerical methods for digital computers, and advanced engineering mathematics.

Swedish Board for Computing Machinery.—The Board for Computing Machinery was appointed by the Swedish Government in November 1948. At present its members are:

Rear Admiral G. JEDEUR-PALMGREN, Stockholm
Professor TORBERN LAURENT, Stockholm
Professor EDY VELANDER, Stockholm
Permanent Secretary GUSTAV ADOLF WIDELL, Stockholm (*chairman*)
Professor NILS ZEILON, Lund.

The Board has the authority to make decisions concerning the development and acquisition of computing machinery for the Swedish State within a budgetary frame fixed by the Parliament. Its aim is to provide computational service to Swedish State agencies as well as to private institutions and enterprises.

The following persons have been appointed by the Board to act in an advisory capacity: Mr. GUNNAR BERGGREN, Stockholm, Professor STIG EKELÖF, Gothenburg, Dr. CARL-ERIK FRÖBERG, Lund, and Commodore SIGURD LAGERMAN, Stockholm.

Under the supervision of the Board, Dr. CONNY PALM, Stockholm, is head of the Projects Group, within which the experimental, design, and construction work is carried out. The secretary of the Board is Mr. GÖSTA MALMBERG, Stockholm.

The Board will issue from time to time communications in English, which will be sent to interested persons in other countries. Anyone desiring to receive these communications should write to:

Matematikmaskinnämnden
Drottninggatan 95 A
Stockholm 6.

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z–XIV

14. R. BEREIS, "Mechanismen zur Verwirklichung der Joukowsky-Abbildung," *Archiv für Math.*, v. 2, 1949–1950, p. 126–134.

The author considers two mechanisms which transform the $z = x + iy$ plane into itself according to the JOUKOWSKY transformation. As an introduction, some of the well known properties of the Joukowsky transformation are summarized. By appropriate use of his mechanisms, the author shows how these transformation properties may be realized. The mechanisms discussed consist of modifications of two types of inversors. First, the author considers two PEAUCELLIER inversors, each of which consists of a rhombus linkage. By cross-knotting these inversors, the author obtains the desired mechanism (called a "Zwillings-inversors"). The second mechanism is built up from the HART inversor. This inversor consists of an antiparallelogram.

By adjoining to this mechanism an ordinary parallelogram linkage, the author obtains the second mechanism.

N. COBURN

Univ. of Michigan
Ann Arbor, Mich.

15. Z. W. BIRNBAUM & H. S. ZUCKERMAN, "A graphical determination of sample size for Wilks' tolerance limits," *Ann. Math. Stat.*, v. 20, 1949, p. 313–316.

This note contains a graph for the solution of the equation $N\beta^{N-1} - (N-1)\beta^N = 1 - \epsilon$ for N, given ϵ and β.

F. J. M.

16. A. R. BOOTHROYD, E. C. CHERRY & R. MAKAR, "An electrolytic tank for the measurement of steady-state response, transient response, and allied properties of networks," Inst. Elect. Eng., *Proc.*, v. 96, sect. 1, 1949, p. 163–177.

Suppose that an infinite thin sheet of conducting material corresponds to the plane of the complex variable λ and that electrodes carrying equal currents are placed at points $\mu_1, \cdots, \mu_k$ and electrodes with equal but oppositely directed currents are placed at $\gamma_1, \cdots, \gamma_l$. Then, except for certain scale constants, the potential $V(\lambda)$ is given by

$$V(\lambda) = \sum \log|\lambda - \mu_i| - \sum \log|\lambda - \gamma_i| = \log|Z(\lambda)|$$

where

$$Z(\lambda) = \frac{(\lambda - \mu_1) \cdots (\lambda - \mu_k)}{(\lambda - \gamma_1) \cdots (\lambda - \gamma_l)}.$$

The conjugate to the harmonic function $V(\lambda)$ is the phase angle of $Z(\lambda)$ and this represents a powerful method of representing the rational function $Z(\lambda)$.

The authors discuss various applications of this analogy and present two methods for eliminating the difficulty inherent in the requirement that the sheet be infinite in extent. In one of these, two circular sheets of conducting material are joined on their outer rim. One circular sheet corresponds to a circle around the origin in the complex plane, the other circular sheet corresponds to the rest of the complex plane with the point at infinity in the middle of the circular sheet. The other solution to the difficulty takes advantage of the fact that in the majority of applications, the roots and poles of $Z(\lambda)$ are symmetrically placed relative to the real axis so that no current flows across this axis in the analogue. Consequently attention can be confined to either upper or lower half plane, and this in turn can be conformally mapped on a finite circle.

A variety of applications are discussed in the paper, especially those associated with an impedance function $Z(\lambda)$. For instance, such an analogue can be used to find the roots of a polynomial $f(\lambda)$ by the method of LUCAS. If f is of the n-th degree, a polynomial $g(\lambda)$ with $n + 1$ known real roots is introduced in order to form the rational function $F(\lambda) = f(\lambda)/g(\lambda)$. Since the roots of $g(\lambda)$ are real, the residues of $F(\lambda)$ at the corresponding poles are real and

$$F(\lambda) = \sum a_j(\lambda - \alpha_j)^{-1}$$

where a_j are real. Electrodes are set up at the roots α_j of $g(\lambda)$ and fed currents of strength a_j. The potential function is then, if $\lambda = x + iy$, $\alpha_j = \alpha_j' + i\alpha_j''$,

$$V(x, y) = \sum a_j \log|\lambda - \alpha_j| = \sum a_j \tfrac{1}{2} \log\left((x - \alpha_j')^2 + (y - \alpha_j'')^2\right).$$

The roots of $f(\lambda) = 0$ which are also those of $F(\lambda) = 0$ are located by finding the points where both $\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial V}{\partial y}$ are zero. For

$$\frac{\partial V}{\partial x} = \sum a_j \frac{x - \alpha_j'}{(x - \alpha_j')^2 + (y - \alpha_j'')^2} = \sum a_j \frac{x - \alpha_j'}{(\lambda - \alpha_j)(\bar{\lambda} - \bar{\alpha}_j)}$$

and similarly

$$\frac{\partial V}{\partial y} = \sum a_j \frac{y - \alpha_j''}{(\lambda - \alpha_j)(\bar{\lambda} - \bar{\alpha}_j)}.$$

From these, one readily shows that $F(\lambda) = \dfrac{\partial V}{\partial x} - i\dfrac{\partial V}{\partial y}$ and thus the simultaneous vanishing of $\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial V}{\partial y}$ yields a root. The paper also describes the methods used to find the phase angle of $Z(\lambda)$ as represented above and the residues of $Z(\lambda)$ at poles.

Two models of the double sheet representation have been set up in the form of electrolytic tanks with a glass insulating disk for separating the electrolyte into two sheets. (A correction for the thickness of the electrolyte is described.) An illustration shows five electrodes and a probe. In addition a second version having fixed probes along the frequency axis is mentioned. The voltages from these fixed probes are measured in rapid succession by a stepping relay and displayed on a long persistence cathode ray tube in order to plot frequency characteristics of an impedance function $Z(\lambda)$.

F. J. M.

17. E. C. Cherry, "The analogies between the vibrations of elastic membranes and the electromagnetic fields in guides and cavities," Inst. Elec. Eng., *Proc.*, v. 96, part III, 1949, p. 346–360.

In addition to the analogies mentioned in the title, the use of lumped circuit elements is also discussed.

F. J. M.

18. E. A. Goldberg, "Stabilization of wide-band direct-current amplifiers for zero and gain," R. C. A. *Rev.*, v. 11, 1950, p. 296–300.

In analogue computers, d.c. feedback amplifiers are used both as integrating and summing amplifiers. The major problem in these has always been drift. In the present paper a method of stabilizing a d.c. amplifier is described which does not adversely affect the frequency characteristics. The summing point voltage of a direct coupled feedback amplifier is chopped and the result is a.c. amplified, rectified and fed back to a zero set point of the direct coupled amplifier. It is shown that this prevents drift and variation of gain due to drift, and for low frequencies yields a very high loop gain.

F. J. M.

19. C. E. Grosser, "Involute-gear geometry. Nondimensional analysis and design of tooth forms, tooth thickness, and mesh relationships." A.S.M.E., *Trans.*, v. 71, 1949, p. 535–554.

The basic properties of the involute permit the design of involute gears based on the equivalence of their action to a crossed belt drive. The author constructs a "belt-length-ratio" chart "to simplify the computations necessary for the analysis of involute gears and for design for optimum performance."

F. J. M.

20. J. F. Heiss & James Coull, "Nomographs speed flow calculation," *Chem. Eng.*, v. 56, April 1949, p. 104–107.

"Three charts are presented to facilitate determining flow rate or time of discharge from a tank through a pipe."

21. J. A. Hrones & J. B. Reswick, "The electronic analogue—a design tool," *Machine Design*, v. 21, Sept. 1949, p. 115–124.

The electronic analogue is a device which produces repeating solutions of the problem under investigation at the rate of sixty per second; these solutions may be conveniently viewed by oscillographic means.

The analogue is essentially a linear device, but non-linearities of one variable of a special type (similar to saturation effects) may also be simulated. The small amplitude response of systems non-linear in one or two variables may be determined quickly by exploring over the range of the parameters.

A typical servo system is chosen as an example to be analyzed by the electronic analogue. The block diagram of the servo system is transformed to a form suitable for setting the problem up on the analogue. The equivalence between mechanical systems and electrical circuits is indicated. Solutions which were obtained from the analogue of output angle, angular error, and motor torque versus time for different types of disturbances, and different system gain settings, are indicated. The effect of non-linear motor characteristics is also demonstrated.

The article demonstrates how the electronic analogue can be useful in understanding and analyzing the performance of a servo system, as a specific example.

Sidney Godet

Reeves Instrument Corp.
New York, N. Y.

22. C. F. Kayan, "Heat-transfer temperature patterns of a multicomponent structure by comparative methods," A.S.M.E., *Trans.*, v. 71, 1949, p. 9–16.

A complex problem in heat flow is solved in three different ways. In one of these, an electrical analogue involving slabs of conducting materials is used; in the second an electrical network, and in the third a computational network procedure. Greatest accuracy is claimed for the first method, while the second is considered to be the most convenient when appropriate equipment is available.

F. J. M.

23. D. B. LANGMUIR, "An automatic plotter for electron trajectories," *R. C. A. Rev.*, v. 11, 1950, p. 143–154.

The electric potential is simulated in an electrolytic tank, and the gradient of this potential and the velocity of the electron determine the radius of curvature of the path. The radius of curvature is used to obtain a mechanical plot of the path. A detailed description of the technical difficulties in this set up is given.

F. J. M.

24. G. D. MCCANN & C. H. WILTS, "Application of electric-analog computers to heat-transfer and fluid-flow problems," *Jn. App. Mech.*, v. 16, 1949, p. 247–258, bound with v. 71, A.S.M.E., *Trans.*

A brief description of the Caltech electric analog computer is given which contains a list of available components [*MTAC*, v. 3, p. 501–513]. The analogy upon which certain heat flow problems have been solved is described. Certain of these involve ordinary differential equations with step function coefficients, in others a grid pattern is used to solve a non-linear partial differential equation, linear in the partial derivatives but with coefficients dependent upon the unknown function.

F. J. M.

25. J. K. MICKELSEN, "Automatic equipment and techniques for field mapping," *Gen. Elect. Rev.*, v. 52, no. 11, Nov. 1949, p. 19–23.

Equipment for plotting the equipotential lines in an electrolytic tank is described. Problems of either the LAPLACE or POISSON type may be considered. Three electrodes, set up in a straight line are positioned by servomotors in such a way that their line is approximately tangent to an equipotential line. Thus the slope of the line is determined and the electrode assembly is moved along this line. The relative amount of x and y motion is obtained from this slope by means of resolvers. However there is an additional servo signal which permits a more accurate positioning of the electrode assembly and also permits one to choose the potential of the equipotential line that is being plotted.

F. J. M.

26. K. I. TIKHOTSKIĬ, "Nomogrammy dli͡a vychislenii͡a opredeliteleĭ Gurvitsa" (Nomograms for the calculation of Hurwitz determinants), *Avtomatika i Telemekhanika*, v. 9, 1948, p. 152–155 + one folding plate.

The real parts of the roots of $a_0 + a_1x + \cdots + a_nx^n = 0$ are negative if and only if[1] $a_0 > 0, \Delta_1 > 0, \cdots, \Delta_n > 0$ where Δ_m is the Hurwitz determinant

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots 0 \\ a_3 & a_2 & a_1 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_{2m-1} & a_{2m-2} & a_{2m-3} & a_m \end{vmatrix}$$

in which $a_k = 0$ for $k > n$. The author has constructed charts to evaluate the Δ_m for equations up to and including degree 6. In the paper they are reproduced for Δ_2 (deg 3), Δ_3 (deg 4), Δ_3 (deg 5) and in part Δ_4 (deg 6). The

construction and arrangement of the charts can be understood by explaining the case of Δ_4. For degree 5 it is found that

$$\Delta_4 = a_4\Delta_3 - a_5B$$

where

$$\Delta_3 = a_3\Delta_2 - a_1A, \qquad B = a_2\Delta_2 - a_0A$$

and

$$\Delta_2 = a_1a_2 - a_0a_3, \qquad A = a_1a_4 - a_0a_5.$$

Each of these relations is represented by a chart with three parallel scales, two of which are binary (entered by perpendicular projection from corresponding triangular networks). These charts are in part superimposed along the common scales.

R. CHURCH

U. S. Naval Postgraduate School
Annapolis, Md.

[1] A. HURWITZ, "Über die Bedingungen unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt," *Math. Ann.*, v. 46, 1895, p. 273–284; *Math. Werke*, v. 2, p. 533–545.

NOTES

121. PRODUCTION OF TABLES OF MULTIPLICATIVE FUNCTIONS BY PUNCHED CARD EQUIPMENT.—Numerical functions $f(n)$ for which the functional equation

$$(1) \qquad f(m)f(n) = f(mn)$$

holds for every pair (m, n) of relatively prime integers are called multiplicative and constitute a conspicuous class of functions. Examples are EULER's totient function $\phi(n)$ (enumerating the numbers not exceeding n and prime to n), the sum $\sigma(n)$, and the number $\nu(n)$ of divisors of n, extensive tables of which were computed by GLAISHER.[1] The purpose of this note is to point out that punched card tables of such functions can be produced easily by means of IBM equipment consisting of the sorter, the collator and any one of the 600 type machines.

The most general solution of the equation (1) is obtained by assigning arbitrarily the values of $f(p^\alpha)$ for every prime p and every positive integer α. The value of $f(n)$ for $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ is then defined by $f(n) = f(p_1^{\alpha_1}) \cdots f(p_t^{\alpha_t})$. Hence one begins by producing a table of $f(p^\alpha)$, where f is the given function. This may present some difficulties in case $f(p^\alpha)$ is a complicated function of p and α. Indeed in some cases one may be stopped by the fact that $f(p)$ is an unknown function of the prime p. This occurs for example in the case of RAMANUJAN's function $\tau(n)$. In many other cases, however, $f(p^\alpha)$ is a polynomial in p^α or some equally simple function which can be computed on a 600 type machine, or otherwise.

Let us suppose that we wish to produce a table of $f(n)$ for $n = 1(1)N$. Let p_k be the greatest prime less than $N^{\frac{1}{2}}$. The table is constructed in various steps as follows.

We begin with the set $S(p_k)$ of cards punched with the values

$$n, f(n)$$

where $n = p$ with $p \leqslant N$ and $p > p_k$; in this set we include also the card for $n = 1(f(1) = 1)$. The set of cards is arranged according to increasing values of n. From the set $S(p_k)$ we select the set $S'(p_k)$ of cards for which $n \leqslant N/p_k$. With the collator we interfile a blank card between each member of $S'(p_k)$. This deck of cards is now inserted in the hopper of a 600 type machine, which multiplies n by p_k and $f(n)$ by $f(p_k)$ and punches these values into the blank card following the card corresponding to n. The fields for this punching are of course the same as those of the cards already produced. The output of this operation is returned to the collator which now "demerges,"[2] i.e., separates out those cards which have just been punched from the cards of the original set $S'(p_k)$. The former cards are then filed in order among the cards of $S(p_k)$. The latter cards are also returned to the set $S(p_k)$. Finally the card for $n = p_k^2$ is also filed in the set $S(p_k)$. The result is a set $S(p_{k-1})$ of cards corresponding to all integers $n \leqslant N$ divisible by no prime less than p_k.

The set $S(p_{k-1})$ is dealt with in the same manner to produce $S(p_{k-2})$ and so on. Eventually, however, the newly punched cards become too numerous to file with efficiency in the main set $S(p_n)$, and the sorter is used instead to arrange the cards of $S(p_{n-1})$ in their proper order. Also as soon as we descend to the first prime p_n for which $p_n \leqslant N^{\frac{1}{2}}$, it will be necessary to make a (small) additional run multiplying by p_n^2 and $f(p_n^2)$ and to include in $S(p_{n-1})$ the card for $n = p_n^2$.

Continuing the process we have in the final step a table of $f(n)$ for all odd values of $n \leqslant N$. This is the set $S(2)$. The first half of this set $(S'(2))$ is multiplied by $f(2)$, the first quarter by $f(2^2)$, etc. Finally the cards for $n = 2^{\alpha-1}$ and $n = 2^\alpha$, where $2^\alpha \leqslant N < 2^{\alpha+1}$ are filed in to complete the table. From time to time a card count by the sorter and a sequence check by the collator on the argument n is a good precaution.

Finally, if the table is listed on a tabulator, the sums

$$\sum_{n \leq x} f(n)$$

may be obtained for several values of $x \leqslant N$ without extra effort. These may be compared with predicted values, which, in most cases of f, are obtainable from theoretical considerations, in order further to check the table.

The time required to complete the table, once the cards for $f(p^\alpha)$ are produced, is governed by the time required by the 600 type machine, since the collator and sorter are relatively fast and can be operated during the multiplication process after the whole procedure is started.

The writer constructed a table of $\sigma_5(n)$, the sum of the fifth powers of the divisors of n for $n = 1(1)5000$ by this method, using a 602A calculating punch. [See UMT 114.]

D. H. L.

[1] J. W. L. Glaisher, *Number-Divisor Tables* (BAASMT, v. 8), Cambridge, 1940.

[2] This demerging can also be done on the sorter if the 600 type machine is programmed to emit a distinguishing punch in an otherwise unused column of the newly punched card, this punch being characteristic of the particular multiplication program. This scheme is useful also in case it is found, at some later time, that mistakes were made at a particular stage of the process. These cards and all other cards affected by these mistakes can then be sorted out for correction.

122. Correction to the article, "Matrix Inversion by a Monte Carlo Process."—In the proof of Theorem 1 of the above article [*MTAC*, v. 4, p. 127–129] it was tacitly assumed that the sum given for $E(G_{ij})$ was absolutely convergent, since otherwise the first absolute moment of G_{ij} and therefore $E(G_{ij})$ fail to exist. We must therefore replace assumption (L) of the article by a stronger hypothesis, namely

$$\max_r |\lambda_r(A^*)| < 1,$$

where A^* is the matrix with non-negative elements $|A_{ij}|$.

George E. Forsythe
Richard A. Leibler

NBSINA, Univ. of Calif., Los Angeles
2901, 18th St., Washington, D. C.

123. On the number $2^{161} + 1$.—I have made a study of the number

$$N = (2^{161} + 1)/3$$

with a view of establishing its prime or composite character. A search for a prime factor less than $6 \cdot 10^6$ was unsuccessful. On the other hand if N were a prime we should have

$$3^{3N-1} \equiv 9 \pmod{N}.$$

Actually, I find

$$3^{3N-1} \equiv 54302\ 73773\ 60852\ 63755\ 11740\ 55612\ 78194\ 90019\ 88969 \pmod{N}.$$

Hence N is composite.

A. Ferrier

Collège de Cusset
Allier, France

QUERIES

36. Exponential Integrals for Complex Argument.—Are there tables of the integrals

$$\int_x^\infty t^{-1}e^{-at}\cos t\,dt, \qquad \int_x^\infty t^{-1}e^{-at}\sin t\,dt,$$

or of related functions from which these integrals may be evaluated? The parameters a and x are positive.

G. C. Tibbetts

Tibbetts Laboratories
Camden, Maine

CORRIGENDA

V. 4, p. 156, l. 22, p. 251, *for* Arenburg *read* Arenberg.
V. 4, p. 179, l. 2, *for* $C = 2$ *read* $C = -2$.
V. 4, p. 180, l. -14, *for* 54 *read* 554.
V. 4, p. 238, l. -11, p. 256, *for* P. A. Morton *read* P. L. Morton